Journal of Statistical Physics, Vol. 105, Nos. 3/4, November 2001 (© 2001)

Gibbs Measures for Brownian Paths Under the Effect of an External and a Small Pair Potential

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Received April 9, 2001

We consider Brownian motion in the presence of an external and a weakly coupled pair interaction potential and show that its stationary measure is a Gibbs measure. Uniqueness of the Gibbs measure for two cases is shown. Also the typical path behaviour, the degree of mixing and some further properties are derived. We use cluster expansion in the small coupling parameter.

KEY WORDS: Brownian motion; Gibbs measures; cluster expansion.

1. INTRODUCTION

Gibbs measures are well-studied objects for models of discrete state space or point processes in the continuum. For such Gibbs measures a number of powerful techniques are available, among them cluster expansion methods.

There are cases where the understanding of Gibbs measures becomes important in a more complicated set-up. One example is the investigation of certain quantum systems by using the Feynman–Kac formula. This approach makes possible to describe the properties of specific quantum systems in terms of Gibbs measures associated with stochastic processes defined on path space. In the simplest case of a single non-relativistic quantum particle placed under a potential this approach reduces to studying so called $P(\phi)_1$-processes, i.e., a particular class of Markov processes in $\mathbb{R}^d$.[1, 17]

The present work is motivated by similar problems, specifically the investigation of systems consisting of a quantum particle interacting with a

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0022-4715/01/1100-0605$19.50/0

quantum field. Perhaps the simplest such case is Nelson's model with the following action (in its so called "Euclidean form;" see refs. 2 and 12 for details):

$$S(\{q_t, \phi_t\}) = \int_{\mathbb{R}^d} \int (\dot{\phi}_t^2(x) + \nabla\phi_t(x)^2)\, dx\, dt + \int (\dot{q}_t^2 + V(q_t))\, dt$$

$$+ e \int_{\mathbb{R}^d} \int \phi_t(x)\, \rho(q_t - x)\, dx\, dt. \tag{1.1}$$

Here $\phi_t(x)$ represents the Euclidean quantum field on space-time $\mathbb{R}^d \times \mathbb{R}$, $\{q_t\}$ is a path of the particle in $\mathbb{R}^d$, V is the external potential acting on the particle, $\rho > 0$ is the "smeared" charge distribution (with $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$), and e is the coupling constant. Formally, the probability measure on the joint paths $\{q_t, \phi_t\}$ is given by the Feynman–Kac formula:

$$d\mathscr{P} \text{ “}=\text{” } \frac{1}{Z} e^{-S(q_t, \phi_t(x))} \prod_{x,t} d\phi_t(x) \prod_t dq_t. \tag{1.2}$$

This formula turns out to be very useful in studying the ground state of the quantum system, but to give first a mathematically meaningful sense to it we have to construct the corresponding stationary Markov process on the state space $\Omega = \mathscr{S}'(\mathbb{R}^d) \times \mathbb{R}^d$. In this description then the Hilbert space $L^2(\Omega, d\mathsf{P})$, with $d\mathsf{P}$ the invariant measure for (1.2), is to be the physical space of the interacting system, and the generator of the stochastic semigroup associated with the process is identified as the Hamiltonian of the system. By using the special form of the action given by (1.1) we can compute the probability measure conditional on a fixed path which will appear as a Gaussian measure on $C(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$. Then by integrating over the field variables we obtain the marginal distribution for paths $\{q_t\}$ heuristically given by

$$d\mu \text{ “}=\text{” } \frac{1}{\text{normalization}} e^{-\int (\dot{q}_t^2 + V(q_t))\, dt} e^{-\iint W(q_t - q_s, s-t)\, ds\, dt} \prod_t dq_t$$

$$= \frac{1}{\mathscr{Z}} e^{-\iint W(q_t - q_s, s-t)\, ds\, dt}\, d\mu^0 \tag{1.3}$$

where $d\mu^0$ is the stationary measure associated with the $P(\phi)_1$-process generated by the potential V, and $\mathscr{Z}$ is the normalizing partition function. Moreover, the pair potential is given in this case by

$$W(q, t) = -\frac{e^2}{8} \int_{\mathbb{R}^d} \frac{|\hat{\rho}(k)|^2}{|k|} e^{-|k|\,|t|} \cos(k \cdot q)\, dk \tag{1.4}$$

($\hat{\rho}$ is the Fourier transform of ρ). Formula (1.3) shows that the relevant measure can be thought of as being a Gibbs measure relative to Brownian motion for external potential V and pair potential W (or, equivalently, simply for W modifying the $P(\phi)_1$-measure for V taken as reference), defined on path space. In order to carry through the Euclidean construction we will need this measure in the infinite-time limit. One of the basic results we obtain in this paper is that this limit Gibbs measure does exist in a specific sense.

Although Nelson's model provides a clear-cut motivation, we look here at the existence and properties of Gibbs measures of the type (1.3) for more general potentials than the one discussed above. This more general perspective may be interesting at least for other possible applications such as in stochastic partial differential equations (see ref. 9) and continuum limits of models of systems of real-valued lattice spins. We choose the pair potential in two ways: One is allowing it to grow at most quadratically in the space component and decay with a power larger than 2 of time, the other is choosing it uniformly bounded in the spatial variable and falling off in time with a power larger than 1. (In particular, the Nelson model is covered by the latter, and its dipole-approximation by the former.) The one-body potential will be assumed sufficiently strong-binding so that the associated Schrödinger operator (i.e., the generator of the underlying $P(\phi)_1$-process) defines an intrinsically ultracontractive semigroup. We think, however, that the conditions (A1) below can be further weakened.

The existence of Gibbs measures on path space has been addressed also in ref. 16, where however the strong restrictions drawn on the potentials made possible the use of correlation inequalities. In our framework, for showing existence of a Gibbs measure this is no longer applicable, furthermore no compactness arguments apply and there are also no obvious candidates of superstability-like conditions around. The technique we develop here instead is cluster expansion in its familiar polymer-expansion form. However, in details we diverge from other existing schemes for in our context the set-up is essentially different from the commonly used models. We divide $\mathbb{R}$ into intervals and break up the Brownian paths into pieces by restricting them to these intervals. The system thus appears as a "spin chain" having as state space the set of Brownian paths on an interval of a given length, and with an a priori distribution as the invariant measure of a $P(\phi)_1$-process. The interaction is provided by a long range pair potential on whose decay we need sufficiently detailed information (see (A2) below), while the one-body interaction is subsumed in the reference process. In this sense our work relates with the problems discussed in refs. 5, 3 and is a technical simplification over the methods used there. While obviously on the whole our arguments follow the general lines of cluster expansion (see, e.g., refs. 6, 19, 14, and 8), the basic notion of cluster and the ways of controlling the estimates and convergence have to be changed essentially. One specific feature in our approach is exploring the ultracontractivity of the reference process. The method presented here allows a fairly complete understanding of the Gibbs measure, beside its existence and uniqueness we also establish the typical path behaviour and its mixing properties (and more).

The paper is organized as follows. In the next section we define Gibbs measures as perturbations with a weakly coupled pair interaction of a $P(\phi)_1$-reference process. For simplicity we use free boundary conditions. In Section 3 we derive the cluster representation of the partition function. Section 4 contains the cluster estimates. In its first subsection we establish the estimates on the weigths of clusters, and in the second we derive another fundamental cluster estimate that will lead up to showing the convergence of the cluster expansion. In Section 5 we look at the typical path behaviour. In the concluding Section 6 we restrict attention to bounded pair potentials and show uniqueness and some further properties of the Gibbs measure (this material being closely related in particular with Nelson's scalar field model).

2. GIBBS MEASURE FOR AN EXTERNAL AND A PAIR POTENTIAL

Let $\mathscr{X} = C(\mathbb{R}, \mathbb{R}^d)$ be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, endowed with the σ-field $\mathscr{A} = \sigma(\pi_t \colon t \in \mathbb{R})$ generated by the point evaluations $\pi_t \colon \mathscr{X} \to \mathbb{R}^d$, $X \mapsto \pi_t(X) = X_t$. $(\mathscr{X}, \mathscr{A})$ is the measurable space we will use in what follows. For a subset $[T_1, T_2] \subset \mathbb{R}$ we put $\mathscr{X}_{[T_1,T_2]} = C([T_1, T_2], \mathbb{R}^d)$ and for the corresponding σ-field we write $\mathscr{A}_{[T_1,T_2]} = \sigma(\pi_t\colon t \in [T_1, T_2]) \subset \mathscr{A}$. For the interval $[-T, T]$ we will use the notations $\mathscr{X}_T$ respectively $\mathscr{A}_T$. The Lebesgue measure on $\mathbb{R}^d$ will be denoted $d\lambda^d$.

Next, consider two functions $V\colon \mathbb{R}^d \to \mathbb{R}$ and $W\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ called respectively external potential and pair interaction potential, subject to the following conditions:

(A1) External potential. V is bounded from below and continuous, moreover $V(x) = a\,|x|^{2s} + o(|x|^{2s})$, with some $s > 1$ and $a > 0$.

(A2) Classes of pair interaction potential. $W(x_1, x_2, t-s) = W(x_1, x_2, |t-s|)$, and it is symmetric with respect to interchanging x_1 with x_2. Moreover,

(A2-1) There is $R > 0$ and $\alpha > 2$ such that

$$|W(x_1, x_2, t-s)| \leqslant R\,\frac{|x_1|^2 + |x_2|^2}{|t-s|^\alpha + 1} \tag{2.1}$$

for every $x_1, x_2 \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$.

(A2-2) There is $R > 0$ and $\alpha > 1$ such that

$$|W(x_1, x_2, t-s)| \leqslant \frac{R}{|t-s|^{\alpha}+1} \tag{2.2}$$

for every $s, t \in \mathbb{R}$ and uniformly in $x_1, x_2 \in \mathbb{R}^d$.

Under assumption (A1) for the associated Schrödinger operator

$$H = -\tfrac{1}{2}\Delta + V(x) \tag{2.3}$$

we have that $C_0^{\infty}(\mathbb{R}^d)$ is a form core on which the operator is essentially self-adjoint and bounded from below. The bottom of the spectrum E_0 of H is a simple eigenvalue, and the corresponding eigenfunction ψ_0 (ground state) is strictly positive. The semigroup $\{\exp(-tH), t \geqslant 0\}$ exists on $L^2(\mathbb{R}^d, d\lambda^d)$, and it is an integral operator with positive, continuous, uniformly bounded kernel $G_t(x, y)$. Moreover, the semigroup is intrinsically ultracontractive meaning the following (see ref. 4). Take the probability measure $dv = \psi_0^2 \, d\lambda^d$ on $\mathbb{R}^d$, and define the isometry j: $L^2(\mathbb{R}^d, dv) \to L^2(\mathbb{R}^d, d\lambda^d)$, $f \mapsto \psi_0 f$. Take now in $L^2(\mathbb{R}^d, dv)$ the operator H_v with Dom $H_v = j^{-1}(\text{Dom } H)$ and

$$H_v f = (j^{-1}(H - E_0)\, j)\, f = \frac{1}{\psi_0}(H - E_0)(\psi_0 f) = -\frac{1}{2}\Delta f - \left(\frac{\nabla\psi_0}{\psi_0}, \nabla f\right)_{\mathbb{R}^d} \tag{2.4}$$

for every $f \in \text{Dom } H_v$. Then the associated semigroup

$$e^{-tH_v} f = \frac{1}{\psi_0}\, e^{-t(H-E_0)} \psi_0 f \tag{2.5}$$

exists for all $f \in L^2(\mathbb{R}^d, dv)$ and $t \geqslant 0$. Intrinsic ultracontractivity of e^{-tH} means that e^{-tH_v} is ultracontractive, i.e., it maps $L^2(\mathbb{R}^d, dv)$ into $L^{\infty}(\mathbb{R}^d, dv)$ continuously, or, equivalently, $\|e^{-tH_v}\|_{2,\infty} < \infty$, $\forall t \geqslant 0$.

Consider now a Markov process with stationary measure $dv = \psi_0^2 \, d\lambda^d$ and transition probability density

$$g_t(x \mid y) = \frac{G_t(x, y)}{\psi_0(x)\, \psi_0(y)\, e^{-E_0 t}} \tag{2.6}$$

Denote the probability distribution of this process by dP, and by $dP_{[T_1, T_2]}$ its restriction to the field $\mathscr{A}_{[T_1, T_2]}$. We take this as reference process for constructing the finite time Gibbs specification $\mu_{[T_1,T_2]}$ on $\mathscr{X}_{[T_1,T_2]}$ with free boundary conditions

$$d\mu_{[T_1,T_2]}(X) = \frac{1}{Z_{[T_1,T_2]}} e^{-\lambda \int_{T_1}^{T_2} \int_{T_1}^{T_2} W(X_s, X_t, s-t)\, ds\, dt}\, dP(X) \tag{2.7}$$

for every $X \in \mathscr{X}_{[T_1,T_2]}$, $T_1 < T_2 \in \mathbb{R}$, and with parameter $\lambda \in \mathbb{R}$. Here $Z_{[T_1,T_2]}$ is the partition function

$$Z_{[T_1,T_2]} = \int e^{-\lambda \int_{T_1}^{T_2} \int_{T_1}^{T_2} W(X_s, X_t, s-t)\, ds\, dt}\, dP(X) \tag{2.8}$$

turning $\mu_{[T_1,T_2]}$ into a probability measure.

Our main result is

Theorem 2.1. Suppose V and W satisfy assumptions (A1) respectively either (A2-1) or (A2-2). Take an arbitrary decreasing sequence $T_1^{(n)}$ and an increasing sequence $T_2^{(n)}$ of real numbers, $T_1^{(n)} < T_2^{(n)}$, such that $T_1^{(n)} \to -\infty$, $T_2^{(n)} \to \infty$, and $|\lambda| \leqslant \lambda^*$ with λ^* small enough. Then the weak local limit $\lim_{n\to\infty} \mu_{[T_1^{(n)}, T_2^{(n)}]} = \mu$ exists and it is a Gibbs probability measure on $(\mathscr{X}, \mathscr{A})$. Moreover, μ does not depend on the sequences $T_1^{(n)}$, $T_2^{(n)}$.

3. CLUSTER EXPANSION FOR THE PARTITION FUNCTION

For simplicity, from now on we take the symmetric interval $[-T, T]$, and simplify the corresponding subscripts to T. Take a division of $[-T, T]$ into disjoint intervals $\tau_k = (t_k, t_{k+1})$, $k = 0,..., N-1$, with $t_0 = -T$ and $t_N = T$, each of length b, i.e., fix $b = 2T/N$; for convenience we choose N to be an even number so that the origin is endpoint to some intervals. We break up a path X into pieces X_{τ_k} by restricting it to τ_k. The total energy contribution of the pair interaction can be written in terms of the sum

$$W_T := \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, s-t)\, ds\, dt = \sum_{0 \leqslant i < j \leqslant N-1} W_{\tau_i, \tau_j} \tag{3.1}$$

where with the notation $\mathscr{I}_{ij} = \int_{\tau_i} dt \int_{\tau_j} W(X_s, X_t, s-t)\, ds$ we have

$$W_{\tau_i,\tau_j} = \begin{cases} \mathscr{I}_{ij} + \mathscr{I}_{ji} & \text{if } |i-j| \geqslant 2 \\ \frac{1}{2}(\mathscr{I}_{ii} + \mathscr{I}_{jj}) + \mathscr{I}_{ij} + \mathscr{I}_{ji} & \text{if } |i-j| = 1, \text{ and } i \neq 0,\ j \neq N-1 \\ \mathscr{I}_{ij} + \mathscr{I}_{ji} + \frac{1}{2}\mathscr{I}_{00} & \text{if } i = 0 \text{ and } j = 1 \\ \mathscr{I}_{ij} + \mathscr{I}_{ji} + \frac{1}{2}\mathscr{I}_{N-1\,N-1} & \text{if } i = N-1 \text{ and } j = N-2 \end{cases} \tag{3.2}$$

(For keeping the notation simple we do not make explicit the X dependence of these objects.) The following estimate on the double integrals will be useful later on.

Lemma 3.1. We have

$$\int_{\tau_i} dt \int_{\tau_j} W(X_t, X_s, s-t)\, ds \leqslant \begin{cases} C_1 b \dfrac{\int_{\tau_i} X^2_{\tau_i}(t)\, dt + \int_{\tau_j} X^2_{\tau_i}(s)\, ds}{(|j-i-1|\, b)^\alpha + 1} & \text{(A2-1) case} \\ \dfrac{C_2 b^2}{(|j-i-1|\, b)^\alpha + 1} & \text{(A2-2) case} \end{cases} \tag{3.3}$$

with some $C_1, C_2 > 0$ in each case respectively.

Proof. Take the (A2-1) case; clearly it suffices to consider one term of the sum. We have

$$\int_{\tau_i} dt \int_{\tau_j} \frac{X^2_{\tau_i}(t)}{|t-s|^\alpha + 1}\, ds = \int_{\tau_i} \left(\int_{\tau_j} \frac{ds}{|t-s|^\alpha + 1} \right) X^2_{\tau_i}(t)\, dt \tag{3.4}$$

Furthermore (choosing $i < j$)

$$\begin{aligned} \int_{\tau_j} \frac{ds}{|t-s|^\alpha + 1} &\leqslant \int_{bj}^{b(j+1)} \frac{ds}{|s-(i+1)\, b|^\alpha + 1} \\ &= \int_{b(j-i-1)}^{b(j-i)} \frac{du}{|u|^\alpha + 1} \\ &\leqslant \frac{1}{\alpha - 1} \frac{(1+1/k)^{\alpha-1} - 1}{(b(k+1))^{\alpha-1}} \end{aligned}$$

where $k = j - i$. Since $(1+1/k)^{\alpha-1} - 1 \leqslant C/k$ with some number $C > 0$, we obtain

$$\int_{\tau_j} \frac{ds}{|t-s|^\alpha + 1} \leqslant \frac{C_1 b}{(b(k-1))^\alpha + 1} \tag{3.5}$$

with a suitable number $C_1 > 0$. From this the statement follows. For the case (A2-2) the result is obtained in the same way. ∎

By using (3.1) we have

$$e^{-\lambda W_T} = \prod_{0 \leqslant i < j \leqslant N-1} (e^{-\lambda W_{\tau_i, \tau_j}} + 1 - 1) = 1 + \sum_{\mathscr{R} \neq \varnothing} \prod_{(\tau_i, \tau_j) \in \mathscr{R}} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \qquad (3.6)$$

Here the summation is performed over all non-empty sets of different pairs of intervals, i.e., $\mathscr{R} = \{(\tau_i, \tau_j): (\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'}) \text{ whenever } (i, j) \neq (i', j')\}$.

In order to keep this and the forthcoming summations in hand we need a few more notations. Two distinct pairs of intervals (τ_i, τ_j) and $(\tau_{i'}, \tau_{j'})$ will be called *directly connected* and denoted $(\tau_i, \tau_j) \sim (\tau_{i'}, \tau_{j'})$ if one interval of the pair (τ_i, τ_j) coincides with one interval of the pair $(\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is a collection $\{(\tau_{i_1}, \tau_{j_1}),..., (\tau_{i_n}, \tau_{j_n})\}$ in which each pair of intervals is connected to another through a sequence of directly connected pairs, i.e., for any $(\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'})$ there exists $\{(\tau_{k_1}, \tau_{l_1}),..., (\tau_{k_m}, \tau_{l_m})\}$ such that $(\tau_i, \tau_j) \sim (\tau_{k_1}, \tau_{l_1}) \sim \cdots \sim (\tau_{k_m}, \tau_{l_m}) \sim (\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is called a *contour* and denoted by γ. We denote by $\bar{\gamma}$ the set of all intervals that are elements of the pairs of intervals belonging to a contour γ, and by γ^* the set of time-points of intervals appearing in $\bar{\gamma}$. We call two contours γ_1, γ_2 disjoint if they have no intervals in common, i.e., $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \varnothing$. Clearly, $\mathscr{R}$ can be decomposed into maximal connected components, i.e., disjoint contours: $\mathscr{R} = \{\gamma_1,..., \gamma_r\}$ with $\bar{\gamma}_i \cap \bar{\gamma}_j = \varnothing$, $i \neq j$; $i, j = 1,..., r$.

The sum in (3.6) is then further expanded as

$$\sum_{\mathscr{R} \neq \varnothing} \prod_{(\tau_i, \tau_j) \in \mathscr{R}} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) = \sum_{r \geqslant 1} \sum_{\gamma_1,..., \gamma_r} \prod_{k=1}^{r} \prod_{(\tau_i, \tau_j) \in \gamma_k} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \qquad (3.7)$$

where now summation goes over collections $\{\gamma_1,..., \gamma_r\}$ of disjoint contours.

A collection of consecutive intervals $\{\tau_j, \tau_{j+1}..., \tau_{j+k}\}$, $j \geqslant 0$, $j+k \leqslant N-1$ is called a *chain*. As in the case of contours, $\bar{\varrho}$ and ϱ^* mean the set of intervals belonging to the chain ϱ and the set of time-points in ϱ, respectively. Two chains ϱ_1, ϱ_2 are called disjoint if they have no common time-points, i.e., $\varrho_1^* \cap \varrho_2^* = \varnothing$. Take now a non-ordered set of disjoint contours and disjoint chains, $\Gamma = \{\gamma_1,..., \gamma_r; \varrho_1,..., \varrho_s\}$, with some $r \geqslant 1$ and $s \geqslant 0$. Note that such contours and chains may have common timepoints. We use the notation $\Gamma^* = (\bigcup_i \gamma_i^*) \cup (\bigcup_j \varrho_j^*)$ for the set of all timepoints appearing as beginnings or ends of intervals belonging to some contour or chain in Γ. Also, we put $\bar{\Gamma} = (\bigcup_i \bar{\gamma}_i) \cup (\bigcup_j \bar{\varrho}_j)$ for the set of intervals appearing in Γ through entering some contours or chains. Denote by $\partial^- \gamma$ resp. $\partial^+ \gamma$ the leftmost resp. rightmost timepoints belonging to γ, and the similar objects for ϱ. Γ is called a *cluster* if $\{\gamma_1^*,..., \gamma_r^*; \varrho_1^*,..., \varrho_s^*\}$ is a connected collection of sets and for every $\varrho \in \Gamma$ we have that $\partial^- \varrho_l, \partial^+ \varrho_l \in \bigcup_{j=1}^r \gamma_j^*$. This means that in a cluster chains have no loose ends.

Next we fix the positions of path X at the time-points of the division, i.e., we put $X_{t_k} = q_k$, for all $k = 0,..., N$, with $-T = t_0 < t_1 < \cdots < t_N = T$. The distribution of path X in interval $[-T, T]$ conditional on the positions attained at the fixed times is

$$dP_T(X_{\tau_0},..., X_{\tau_{N-1}} \mid X_{t_0} = q_0, X_{t_1} = q_1,..., X_{t_N} = q_N) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k} \mid q_k, q_{k+1}) \tag{3.8}$$

We use the shorthand at the right hand side for the corresponding conditional probabilities for easing the notation. Let $p_{t_0,..., t_N}(q_0,..., q_N)$ be the density with respect to $\prod_{k=0}^{N} d\nu_k(q_k)$ of the joint distribution of positions of path X measured at the time-points $t_0,..., t_N$. Here $d\nu_k$ denotes a copy of $d\nu$ for each $k = 0,..., N$. By the Markov property it then follows that

$$\begin{aligned} p_{t_0,..., t_N}(q_0,..., q_N) &= \prod_{k=0}^{N-1} g_b(q_{k+1} \mid q_k) = \prod_{k=0}^{N-1} (g_b(q_{k+1} \mid q_k) - 1 + 1) \\ &= 1 + \sum_{\mathscr{S}} \prod_{k:\, \tau_k \in \mathscr{S}} (g_b(q_{k+1} \mid q_k) - 1) \end{aligned}$$

The summation is extended over all non-empty sets $\mathscr{S}$ of different pairs of consecutive time-points. In a way similar as before the latter formula can be cast in the form

$$\sum_{\mathscr{S}} \prod_{k:\, \tau_k \in \mathscr{S}} (g_b(q_{k+1} \mid q_k) - 1) = \sum_{s \geqslant 1} \sum_{\varrho_1,..., \varrho_s} \prod_{j=1}^{s} \prod_{k:\, \tau_k \in \varrho_j} (g_b(q_{k+1} \mid q_k) - 1) \tag{3.9}$$

Here $\{\varrho_1,..., \varrho_s\}$ is a collection of disjoint chains, and this formula explains the way we defined them before.

For every cluster $\Gamma = \{\gamma_1,..., \gamma_r; \varrho_1,..., \varrho_s\}$ define the function

$$\kappa_\Gamma = \prod_{l=1}^{r} \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i \tau_j}} - 1) \prod_{m=1}^{s} \prod_{k:\, \tau_k \in \varrho_m} (g_b(q_{k+1} \mid q_k) - 1) \tag{3.10}$$

Also, introduce the auxiliary probability measure on $\mathscr{X}_T$

$$d\mathscr{P}_T(X) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k} \mid q_k, q_{k+1}) \prod_{k=0}^{N} d\nu_k(q_k) \tag{3.11}$$

and look at

$$K_\Gamma = \mathbb{E}_{\mathscr{P}_T}[\kappa_\Gamma] \tag{3.12}$$

Note that $\int (g_b(q_{k+1} \mid q_k)-1)\, d\nu(q_{k+1}) = \int (g_b(q_{k+1} \mid q_k)-1)\, d\nu(q_k) = 0$. This is the reason why from a cluster we rule out chains having loose ends; for any such chain $\mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma] = 0$.

By putting (2.8), (3.7), (3.8), (3.9), (3.10) and (3.12) together we obtain the cluster representation of the partition function:

Proposition 3.2. For every $T > 0$

$$Z_T = 1 + \sum_{n \geqslant 1} \sum_{\{\Gamma_1,\dots,\Gamma_n\}} \prod_{l=1}^{n} K_{\Gamma_l} \tag{3.13}$$

Here the summation is performed over non-ordered collections of clusters $\{\Gamma_1,\dots,\Gamma_n\} \neq \varnothing$ for which $\Gamma_i^* \cap \Gamma_j^* = \varnothing$ whenever $i \neq j$.

As soon as the cluster representation for Z_T is established, the existence of the weak limit measure $\mu := \lim_{T\to\infty} \mu_T$ follows by the cluster estimates below and the general arguments of ref. 13, Chapter 3.

4. CLUSTER ESTIMATES

The first crucial estimate for the cluster expansion is given by

Proposition 4.1. For every cluster Γ we have the bound

$$|K_\Gamma| \leqslant \prod_{\varrho \in \Gamma} (c_1 |\lambda|^{1/3})^{|\bar{\varrho}|} \prod_{\gamma \in \Gamma} \prod_{(\tau_i, \tau_j) \in \gamma} \frac{c_2 |\lambda|^{1/3}}{(|i-j-1|\, b)^\delta + 1} := E_\Gamma(\lambda, \delta) \tag{4.1}$$

with some constants $c_1, c_2 > 0$ and exponent $\delta > 1$.

The details of proof of Proposition 4.1 depend on the particular assumptions we make on the pair interaction potential. We give below the proof for the more complicated case and indicate how it becomes simpler for the other one.

4.1. Proof of Cluster Estimate

We start with potentials V and W satisfying to assumptions (A1) and (A2-1), respectively.

The cluster estimate (4.1) is based on the following generalized version of the Hölder inequality.

Lemma 4.2. Let $(Y_i, \mathscr{Y}_i, \varphi_i)$, $i = 1,..., n$ be a collection of probability spaces and take the product $(Y, \mathscr{Y}, \varphi) = \times_{i=1}^{n} (Y_i, \mathscr{Y}_i, \varphi_i)$. Let $\{\beta_1,..., \beta_m\}$, $\beta_k \subset [1,..., n] := \mathscr{N}_n$, be a collection of subsets of the interval $\mathscr{N}_n$. Suppose $\{f_{\beta_i}, i = 1,..., m\}$ are functions on $(Y, \mathscr{Y}, \varphi)$ measurable with respect to the sub-σ-field $\mathscr{Y}_{\beta_k} = \times_{i \in \beta_k} \mathscr{Y}_i \subset \mathscr{Y}$. Furthermore, let $\{n_{\beta_1},..., n_{\beta_m}\}$ be numbers larger than 1, $i = 1,..., m$, such that for any $i \in \mathscr{N}_n$

$$\sum_{\beta_k:\, i \in \beta_k} \frac{1}{n_{\beta_k}} \leqslant 1 \tag{4.2}$$

Then

$$\left| \int_Y \prod_{j=1}^{m} f_{\beta_k}\, d\varphi \right| \leqslant \prod_{j=1}^{m} \left(\int_{Y_{\beta_k}} |f_{\beta_k}|^{n_{\beta_k}}\, d\varphi_{\beta_k} \right)^{1/n_{\beta_k}} \tag{4.3}$$

with $(Y_{\beta_k}, \varphi_{\beta_k}) = \times_{i \in \beta_k} (Y_i, \varphi_i)$.

For a proof see ref. 6.

First we estimate the integrals in (3.12) for the X_{τ_k} under fixed $q_0,..., q_N$. We take in the lemma above $Y_i = C(\tau_i, \mathbb{R}^d)$, i.e., the space of Brownian paths defined on interval τ_i with the field $\mathscr{A}_{\tau_i} = \mathscr{Y}_i$, and $d\varphi_i = dP_{\tau_i}(\cdot \mid q_i, q_{i+1})$, the conditional distribution on $C(\tau_i, \mathbb{R}^d)$ appearing in (3.11). Then we write $\beta_k = (i, j)$ if $(\tau_i, \tau_j) \in \gamma_l$, $\gamma_l \in \Gamma$, and $f_{\beta_k} = e^{-\lambda W_{\tau_i, \tau_j}} - 1$. Choose $n_{\beta_k} = n_{ij} = A\, |i - j|^{\Delta}$, with some $\alpha > \Delta > 1$ to be specified below, and $A \in \mathbb{N}$ such that

$$\sum_{\beta_k:\, i \in \beta_k} \frac{1}{n_{\beta_k}} \leqslant \sum_{j:\, j \neq i} \frac{1}{n_{ij}} = \frac{1}{A} \sum_{j:\, j \neq i} \frac{1}{|i-j|^{\Delta}} \leqslant \frac{2}{A} \sum_{k=1}^{\infty} \frac{1}{k^{\Delta}} \leqslant \frac{1}{4} \tag{4.4}$$

Then use Lemma 4.2 to get for any $\gamma_l \in \Gamma$, $l = 1, 2,...$

$$\begin{aligned} &\left| \int \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \prod_{i=0}^{N} dP_{\tau_i}(X_{\tau_i} \mid q_i, q_{i+1}) \right| \\ &\quad \leqslant \prod_{(\tau_i, \tau_j) \in \gamma_l} \left(\int |e^{-\lambda W_{\tau_i, \tau_j}} - 1|^{n_{ij}}\, dP_{\tau_i}(X_{\tau_i} \mid q_i, q_{i+1})\, dP_{\tau_j}(X_{\tau_j} \mid q_j, q_{j+1}) \right)^{1/n_{ij}} \end{aligned} \tag{4.5}$$

As a shorthand for

$$\int |e^{-\lambda W_{\tau_i, \tau_j}} - 1|^{n_{ij}}\, dP_{\tau_i}(X_{\tau_i} \mid q_i, q_{i+1}) \mid dP_{\tau_j}(X_{\tau_j} \mid q_j, q_{j+1}) \tag{4.6}$$

we write $F_{n_{i,j}}(q_i, q_{i+1}, q_j, q_{j+1})$ if $j > i+1$, and $F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})$ if $j = i+1$. Thus for $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$,

$$|K_\Gamma| \leqslant \left| \int \prod_{k=0}^{N} dv_k(q_k) \prod_{m=1}^{s} \prod_{\tau_k \in \varrho_m} (g_b(q_{k+1} \mid q_k) - 1) \right.$$
$$\times \prod_{l=1}^{r} \prod_{\gamma_l \in \Gamma} \prod_{(\tau_i, \tau_{i+1}) \in \gamma_l} (F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2}))^{1/n_{i,i+1}}$$
$$\left. \times \prod_{l=1}^{r} \prod_{\substack{(\tau_i, \tau_j) \in \gamma_l \\ j > i+1}} (F_{n_{i,j}}(q_i, q_{i+1}, q_j, q_{j+1}))^{1/n_{ij}} \right| \tag{4.7}$$

follows. Now we apply again Lemma 4.2 to this integral. Take $Y_i = \mathbb{R}^d$, $d\varphi_i = dv$, and pick β_k according to the following classes. Put $\beta_k^{(2)} = \{i, i+1\}$, $\beta_k^{(3)} = \{i, i+1, i+2\}$ and $\beta_k^{(4)} = \{i, i+1, j, j+1\}$, $j > i+1$. Take, moreover, $f_{\beta_k^{(2)}} = g_b(q_{i+1} \mid q_i) - 1$, $f_{\beta_k^{(3)}} = F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})$ and $f_{\beta_k^{(4)}} = F_{n_{i,j}}(q_i, q_{i+1}, q_j, q_{j+1})$. By choosing the corresponding Hölder exponents $n_{\beta_k^{(2)}} = 4$, $n_{\beta_k^{(3)}} = A = n_{j,j+1}$, and $n_{\beta_k^{(4)}} = A\,|j-i|^{\Delta}$, applied with the correct multiplicities we have then by (4.4)

$$\frac{2}{4} + \frac{3}{A} + \sum_{j;\, j > i+1} \frac{4}{A\,|i-j|^{\Delta}} \leqslant \frac{1}{2} + \frac{4}{A} \sum_{k=1}^{\infty} \frac{1}{k^{\Delta}} \leqslant 1 \tag{4.8}$$

so Lemma 4.2 is applicable. We thus obtain

$$|K_\Gamma| \leqslant \prod_{m=1}^{s} \prod_{\tau_k \in \varrho_m} \left(\int |g_b(q_{k+1} \mid q_k) - 1|^4 \, dv_{k+1}(q_{k+1}) \, dv_k(q_k) \right)^{1/4}$$
$$\times \prod_{l=1}^{r} \prod_{(\tau_i, \tau_{i+1}) \in \gamma_l} (F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2}) \, dv_i(q_i) \, dv_{i+1}(q_{i+1}) \, dv_{i+2}(q_{i+2}))^{1/A}$$
$$\times \prod_{\substack{(\tau_i, \tau_j) \in \gamma_l \\ j > i+1}} (F_{n_{ij}}(q_i, q_{i+1}, q_j, q_{j+1}) \, dv_i(q_i) \, dv_{i+1}(q_{i+1}) \, dv_j(q_j) \, dv_{j+1}(q_{j+1}))^{1/n_{ij}} \tag{4.9}$$

We are going to estimate now each factor separately.

Lemma 4.3. For sufficiently large $b > 0$ there exist constants $C > 0$ and $\Lambda > 0$ independent of b such that

$$|g_b(q \mid q') - 1| \leqslant C e^{-\Lambda b} \tag{4.10}$$

uniformly in q, q'.

Proof. As discussed in the previous section, for a potential V satisfying assumption (A1) the associated Schrödinger semigroup $\exp(-tH)$ is intrinsically ultracontractive. By the semigroup property $e^{-bH_\nu} = e^{-H_\nu} e^{-(b-2) H_\nu} e^{-H_\nu}$ we have then that

$$
\begin{aligned}
|g_b(q \mid q') - 1| &= \left| \iint g_1(q \mid x)(g_{b-2}(x \mid y) - 1)\, g_1(y \mid q')\, d\nu(x)\, d\nu(y) \right| \\
&\leqslant N_1^2 \left(\iint (g_{b-2}(x \mid y) - 1)^2\, d\nu(x)\, d\nu(y) \right)^{1/2} \\
&= N_1^2\, e^{-\Lambda(b-2)} \left(1 + \sum_{n=2}^{\infty} e^{-2(E_n - E_1)(b-2)} \right)^{1/2} \\
&\leqslant C e^{-\Lambda b}
\end{aligned}
\tag{4.11}
$$

By taking b large enough, $C > 0$ can be chosen independently of b. Here $N_1 = \|e^{-H_\nu}\|_{2,\infty}$, $E_0 < E_1 \leqslant E_2 \leqslant \cdots$ are the eigenvalues, and $\Lambda = E_1 - E_0$ is the spectral gap of H. We used that the integral $(\iint (g_{b-2}(x \mid y) - 1)^2 \times d\nu(x)\, d\nu(y))^{1/2}$ is the Hilbert-Schmidt norm of $g_{b-2}(x \mid y) - 1$, moreover that the infinite sum $\sum_n e^{-(E_n - E_1)t}$ converges for every $t > 0$; for details see ref. 18. ∎

Next we deal with the second factor in (4.9). We have by (3.3)

$$
\begin{aligned}
&|e^{-\lambda W_{\tau_i, \tau_{i+1}}(X_{\tau_i}, X_{\tau_{i+1}})} - 1|^A \\
&\leqslant (|\lambda|\, |W_{\tau_i, \tau_{i+1}}|)^A\, e^{|\lambda|\, |W_{\tau_i, \tau_{i+1}}|\, A} \\
&\leqslant (c_1 b\, |\lambda|)^A \left(\int_{\tau_i} X_{\tau_i}(t)^2\, dt + \int_{\tau_{i+1}} X_{\tau_{i+1}}(t)^2\, dt \right)^A e^{A c_1 b\, |\lambda|\, (\int_{\tau_i} X_{\tau_i}(t)^2\, dt + \int_{\tau_{i+1}} X_{\tau_{i+1}}(t)^2\, dt)}
\end{aligned}
\tag{4.12}
$$

with some $c_1 > 0$. By using (4.11) we get for sufficiently large b that $|g_b(q \mid q') - 1| \leqslant C e^{-\Lambda b} \leqslant C' < 1$, and thus $g_b(q \mid q') \geqslant C'' > 0$, with some C', C''. Hence we also obtain

$$
p_{t_i, t_{i+1}, t_{i+2}}(q_i, q_{i+1}, q_{i+2}) = g_b(q_{i+1} \mid q_i)\, g_b(q_{i+2} \mid q_{i+1}) > \text{const} \tag{4.13}
$$

where the bound is independent of q_i, q_{i+1}, q_{i+2} and b, and this leads to the estimate

$$
\begin{aligned}
&\int \frac{F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})}{p_{t_i, t_{i+1}, t_{i+2}}(q_i, q_{i+1}, q_{i+2})}\, p_{t_i, t_{i+1}, t_{i+2}}(q_i, q_{i+1}, q_{i+2})\, d\nu(q_i)\, d\nu(q_{i+1})\, d\nu(q_{i+2}) \\
&\qquad \leqslant \text{const}\, (|\lambda|\, b)^A\, \mathbb{E}_P \left[\left(\int_0^{2b} X_t^2\, dt \right)^A e^{A c_1 b\, |\lambda| \int_0^{2b} X_t^2\, dt} \right]
\end{aligned}
\tag{4.14}
$$

Take now the case $j > i+1$ and write for a shorthand $I_j(X) = \int_{\tau_j} X_t^2 \, dt$. By using (3.3) we arrive at

$$
\begin{aligned}
&|F_{n_{ij}}(q_i, q_{i+1}, q_j, q_{j+1})| \\
&\quad \leqslant \int \left(\frac{|\lambda| \, c_1 b (I_i(X) + I_j(X))}{(|i-j-1| \, b)^\alpha + 1} \right)^{n_{ij}} \\
&\qquad \times e^{\frac{c_1 b \, |\lambda| \, n_{ij} (I_i(X) + I_j(X))}{(|i-j-1| \, b)^\alpha + 1}} \, dP_b(X_{\tau_i} \mid q_i, q_{i+1}) \, dP_b(X_{\tau_j} \mid q_j, q_{j+1}) \\
&\quad \leqslant \frac{1}{2} \left(\frac{2 \, |\lambda| \, C_1 b}{(|i-j-1| \, b)^\alpha + 1} \right)^{n_{ij}} \\
&\qquad \times \Big[\int I_i(X)^{n_{ij}} \, e^{|\lambda| \, b I_i(X)} \, dP_b(X_{\tau_i} \mid q_i, q_{i+1}) \int e^{|\lambda| \, b I_j(X)} \, dP_b(X_{\tau_j} \mid q_j, q_{j+1}) \\
&\qquad + \int I_j(X)^{n_{ij}} \, e^{|\lambda| \, b I_j(X)} \, dP_b(X_{\tau_j} \mid q_j, q_{j+1}) \int e^{|\lambda| \, b I_i(X)} \, dP_b(X_{\tau_i} \mid q_i, q_{i+1}) \Big]
\end{aligned} \tag{4.15}
$$

Here we used the inequality for any $x_1, x_2 \geqslant 0$, $n \geqslant 1$

$$(x_1 + x_2)^n \leqslant 2^{n-1}(x_1^n + x_2^n). \tag{4.16}$$

Thus for $j > i+1$ and with the notation

$$D = \max \left\{ A c_1, \frac{c_1 n_{ij}}{(|i-j-1| \, b)^\alpha + 1} = \frac{c_1 A \, |i-j|^\Delta}{(|i-j-1| \, b)^\alpha + 1} \right\} < \infty, \tag{4.17}$$

we obtain

$$
\begin{aligned}
&\int F_{n_{ij}}(q_i, q_{i+1}, q_j, q_{j+1}) \, d\nu(q_i) \, d\nu(q_{i+1}) \, d\nu(q_j) \, d\nu(q_{j+1}) \\
&\leqslant \text{const} \left(\frac{2 \, |\lambda| \, c_1 b}{(|i-j-1| \, b)^\alpha + 1} \right)^{n_{ij}} \mathbb{E}_P \left[\left(\int_0^b X_t^2 \, dt \right)^{n_{ij}} e^{|\lambda| \, Db \int_0^b X_t^2 \, dt} \right] \mathbb{E}_P [e^{|\lambda| \, Db \int_0^b X_t^2 \, dt}]
\end{aligned} \tag{4.18}
$$

The constant prefactor comes about by dividing and multiplying with the densities as in the estimate leading up to (4.14).

We now turn to estimating the P-averages appearing above. Consider the complex function h with parameter $m \in \mathbb{R}$

$$h(z; m) = \mathbb{E}_P [e^{z \int_0^m X_t^2 \, dt}] = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, \mathbb{E}_P \left[\left(\int_0^m X_t^2 \, dt \right)^n \right] \tag{4.19}$$

for $z \in \mathbb{C}$. We have

$$\mathbb{E}_P\left[\left(\int_0^m X_t^2\,dt\right)^n\right] \leqslant m^{n-1}\mathbb{E}_P\left[\int_0^m X_t^{2n}\,dt\right]$$

$$= m^n \int_{\mathbb{R}^d} |x|^{2n}\,d\nu(x)$$

$$\leqslant m^n c \int_{\mathbb{R}^d} |x|^{2n}\,e^{-\epsilon\,|x|^{s+1}/(s+1)}\,d\lambda^d(x)$$

$$= \frac{2m^n C_d}{s+1}\left(\frac{s+1}{\epsilon}\right)^{(2n+d)/(s+1)} \Gamma\left(\frac{2n+d}{s+1}\right) \tag{4.20}$$

Here $C_d > 0$ is a constant dependent on the dimension coming from integrating over $d-1$ angular coordinates, and in the inequality above we used the bound for the ground state

$$|\psi_0(x)|^2 \leqslant c e^{-\epsilon\,|x|^{(s+1)/(s-1)}} \tag{4.21}$$

with appropriate constants $c, \epsilon > 0$, see ref. 6. From here and an application of Stirling's formula to the Gamma-function it follows that $h(z; m)$ is an entire function of order $(s+1)/(s-1)$,[11] and

$$|h(z; m)| \leqslant c_1 e^{c_2 (m\,|z|)^{(s+1)/(s-1)}} \tag{4.22}$$

with appropriate constants $c_1, c_2 > 0$. Thus

$$\mathbb{E}_P[e^{|\lambda|\,Db \int_0^{kb} X_t^2\,dt}] \leqslant c_1 e^{c_2 (kb^2\,|\lambda|\,D)^{(s+1)/(s-1)}} \tag{4.23}$$

for both $k=1$ and $k=2$. By (4.22) we get also

$$\mathbb{E}_P\left[\left(\int_0^{kb} X_t^2\,dt\right)^n e^{|\lambda|\,Db\int_0^{kb} X_t^2\,dt}\right] = \left(\frac{d^n}{dz^n}\,h(z; kb)\right)_{z=|\lambda|\,Db}$$

$$= \frac{n!}{2\pi i}\int_{|\varpi - |\lambda|\,Db| = b\sqrt{|\lambda|}} \frac{h(\varpi; kb)}{(|\lambda|\,Db - \varpi)^{n+1}}\,d\varpi$$

$$\leqslant \frac{c_1 n!}{(|\lambda|\,b^2)^{n/2}}\,e^{c_2 (kb(|\lambda|\,Db + b\sqrt{|\lambda|}))^{(s+1)/(s-1)}}$$

$$\leqslant \frac{c_1 n!}{(|\lambda|\,b^2)^{n/2}}\,e^{\bar{c}_2 (b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}} \tag{4.24}$$

for both $k=1$ and $k=2$. In the latter bound we used that $|\lambda|$ is small enough. Combining (4.14) with (4.24) we find (since $b>1$)

$$\left(\int F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})\, dv(q_i)\, dv(q_{i+1})\, dv(q_{i+2})\right)^{1/A} \leqslant B\sqrt{|\lambda|}\, e^{\bar{c}_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}} \tag{4.25}$$

where $B>0$ is a constant. Similarly, by piecing together (4.18), (4.23) and (4.24) we find

$$\begin{aligned}
&\left(\int F_{n_{ij}}(q_i, q_{i+1}, q_j, q_{j+1})\, dv(q_i)\, dv(q_{i+1})\, dv(q_j)\, dv(q_{j+1})\right)^{1/n_{ij}} \\
&\quad\leqslant \frac{c_1\sqrt{|\lambda|}\,(n_{ij}!)^{1/n_{ij}}}{(|i-j-1|\,b)^{\alpha}+1}\, e^{c_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}} \\
&\quad\leqslant \bar{B}b^2\sqrt{|\lambda|}\,\frac{|i-j|^{\Delta}}{(|i-j-1|\,b)^{\alpha}+1}\, e^{c_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}}
\end{aligned} \tag{4.26}$$

with some constant $\bar{B}>0$. Take $\Delta>1$ such that $\delta:=\alpha-\Delta>1$. Finally, choose

$$b = \frac{1}{3\Delta}\log\frac{1}{|\lambda|} \tag{4.27}$$

From Lemma 4.3 we then have

$$\begin{aligned}
\prod_{m=1}^{s}\prod_{\tau_k\in\varrho_m}\left(\int |g_b(q_{k+1}\mid q_k)-1|^4\, dv_{k+1}(q_{k+1})\, dv_k(q_k)\right)^{1/4} &\leqslant \prod_{m=1}^{s}\,(Ce^{-\Delta b})^{|\bar{\varrho}_m|} \\
&\leqslant \prod_{m=1}^{s}\,(c_1\,|\lambda|^{1/3})^{|\bar{\varrho}_m|}
\end{aligned} \tag{4.28}$$

with $|\bar{\varrho}_m|$ denoting the number of intervals $\tau_k\in\varrho_m$. Thus moreover $b^2\sqrt{|\lambda|}\leqslant|\lambda|^{1/3}$ for small $|\lambda|$. This and estimates (4.25), (4.26) and (4.28) combined complete then the proof of Proposition 4.1. ∎

For pair potentials of type (A2-2) the proof becomes simpler. In particular, Lemma 4.3 holds unchanged, and the estimates following (4.9) become more straightforward as the pair potential is uniformly bounded in the path.

4.2. Convergence of the Cluster Expansion

The second fundamental estimate leading up to the proof of Theorem 2.1 and based on (4.1) is

Proposition 4.4. There is a constant $c > 0$ independent of λ, and a number $0 < \eta(\lambda) < 1$ (with $\eta \to 0$ as $\lambda \to 0$) such that

$$\sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ |\bar{\Gamma}| = n}} |K_\Gamma| \leqslant c\,\eta^n \tag{4.29}$$

Proof. In this section we put for a shorthand

$$\varepsilon = c_1\, |\lambda|^{1/3} \quad \text{and} \quad \mathscr{D}(\gamma) = \prod_{(\tau_i, \tau_j) \in \gamma} \frac{\varepsilon}{|i-j-1|^\delta + 1} \tag{4.30}$$

Consider the complex function

$$H(z; \lambda) = \sum_{\Gamma:\, \Gamma^* \ni 0} K_\Gamma z^{|\bar{\Gamma}|} = \sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ \Gamma \supset \text{ one contour}}} K_\Gamma z^{|\bar{\Gamma}|} + \sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ \Gamma \supset \text{ more than one contour}}} K_\Gamma z^{|\bar{\Gamma}|} \tag{4.31}$$

We show that for sufficiently small $|\lambda| \neq 0$ this is an analytic function of z in a large circle of radius $R(\lambda)$, with $R(\lambda) \to \infty$ as $\lambda \to 0$. Moreover, we show that within this circle $H(z; \lambda)$ is uniformly bounded in λ. This will then imply (4.29) by choosing $\eta(\lambda) = 1/R(\lambda)$.

Let us start by estimating the second sum. Let $\mathscr{V} = \{\gamma_1,..., \gamma_r\}$, $r \geqslant 2$ be a collection of pairwise disjoint contours. We view $\mathscr{V}$ as a fixed vertex set and consider all possible connected graphs $\mathscr{G}$ associated with it. Then by Proposition 4.1 the estimate becomes

$$\begin{aligned} &\Bigg| \sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ \Gamma \supset \text{ more than one contour}}} K_\Gamma z^{|\bar{\Gamma}|} \Bigg| \\ &\quad \leqslant \sum_{r \geqslant 2} \sum_{\{\gamma_1,..., \gamma_r\}} \sum_{\mathscr{G}} \sum_{s \geqslant 0} \\ &\quad \times \sum_{\substack{\{\varrho_1,..., \varrho_s\} \\ \text{compatible with } \mathscr{G} \text{ and } 0 \in \bigcup_i \gamma_i^* \cup \varrho_i^*}} \prod_{i=1}^{r} ((|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_i|}\, \mathscr{D}(\gamma_i) \prod_{j=1}^{s} (\varepsilon\,|z|)^{|\hat{\varrho}_j|} \end{aligned} \tag{4.32}$$

The sum over $\mathscr{G}$ is meant to be taken over all connected graphs with vertex set $\mathscr{V}$ and set of edges $\{(\gamma_i, \gamma_j)\colon \gamma_i, \gamma_j \in \mathscr{G};\, i, j = 1,..., r\}$; connected graphs are those for which either $\gamma_i^* \cap \gamma_j^* = \varnothing$ or there exists $\varrho_l \in \Gamma$ such that $\varrho_l^* \cap \gamma_i^* \neq \varnothing \neq \varrho_l^* \cap \gamma_j^*$. We denoted by $\{\hat{\varrho}_1,..., \hat{\varrho}_s\}$ the collection of chains satisfying to the conditions below:

1. for every pair $(\gamma_i, \gamma_j) \in \mathscr{G}$, $\gamma_i^* \cap \gamma_j^* = \varnothing$, there is at least one chain $\hat{\varrho}_l$ of this collection connecting γ_i and γ_j (i.e., $\bar{\hat{\varrho}}_l \cap \bar{\gamma}_i = \varnothing = \bar{\hat{\varrho}}_l \cap \bar{\gamma}_i$ and $\hat{\varrho}_l^* \cap \gamma_i^* \neq \varnothing \neq \hat{\varrho}_l^* \cap \gamma_j^*$), and for any pair $(\gamma_i, \gamma_j) \notin \mathscr{G}$ such a chain does not occur;
2. the sets $\bigcup_i \bar{\gamma}_i$ and $\bigcup_j \bar{\hat{\varrho}}_j$ are disjoint, and $\bigcup_j \{\partial^- \hat{\varrho}_j, \partial^+ \hat{\varrho}_j\} \subset \bigcup_i \gamma_i^*$;
3. $0 \in (\bigcup_i \gamma_i^*) \cup (\bigcup_j \hat{\varrho}_j^*)$;
4. each chain $\hat{\varrho}_k$ connects some pair $(\gamma_i, \gamma_j) \in \mathscr{G}$ (we refer to this event by $\gamma_i \leftrightarrow \gamma_j$), or fills a gap within some contour γ_i (referred to as $\gamma_i \leftrightarrow \gamma_i$).

Note that each $\hat{\varrho}$ can join only one pair of contours. The collection $\{\hat{\varrho}_1,..., \hat{\varrho}_s\}$ of chains is then constructed in the following way:

1. first remove all chains $\varrho_l \in \Gamma$ for which $\bar{\varrho}_l \subset \bigcup_{i=1}^r \bar{\gamma}_i$;
2. for all remaining chains $\varrho_k \in \Gamma$, $\bar{\varrho}_k \not\subset \bigcup_{i=1}^r \bar{\gamma}_i$ remove all intervals from the set $\bar{\varrho}_k \cap (\bigcup_{i=1}^r \bar{\gamma}_i)$;
3. of the remaining intervals form all possible collections of non-empty chains denoted by $\{\hat{\varrho}_1,..., \hat{\varrho}_m\}$.

Note that for fixed $\{\gamma_1,..., \gamma_r\}$ the collection $\{\hat{\varrho}_1,..., \hat{\varrho}_m\}$ can be obtained from many possible collections of chains $\{\varrho_1,..., \varrho_s\}$. This fact accounts for the factor $(1+\varepsilon\,|z|)^{|\bar{\gamma}|}$ appearing at the right hand side of (4.32).

Thus we get that for any graph $\mathscr{G}$ with vertex set $\mathscr{V}$

$$\sum_s \sum_{\substack{\{\hat{\varrho}_1,..., \hat{\varrho}_s\} \\ \text{compatible with } \mathscr{G} \text{ and } 0 \in \bigcup_i \gamma_i^* \cup \varrho_i^*}} \prod_{j=1}^{s} (\varepsilon\,|z|)^{|\bar{\hat{\varrho}}_j|}$$

$$\leqslant \prod_{(\gamma_i, \gamma_j) \in \mathscr{G}} f_2(\gamma_i, \gamma_j) \prod_{\gamma_i \in \mathscr{V}} f_1(\gamma_i)(\varepsilon\,|z|)^{\operatorname{dist}(0, \{\gamma\})/3} \tag{4.33}$$

with $\operatorname{dist}(0, \{\gamma_i\}) = \min_i \operatorname{dist}(0, \gamma_i)$, $\operatorname{dist}(0, \gamma) = \min_{\tau \in \gamma} \operatorname{dist}(0, \tau)$, and $\operatorname{dist}(0, \tau_i) = N/2 - i - 1$ if $N/2 - 1 \geqslant i$, and $\operatorname{dist}(0, \tau_i) = i - N/2$ if $N/2 \leqslant i$ (remember, N is even). Moreover

$$f_2(\gamma_i, \gamma_j) = \sum_{\substack{\{\hat{\varrho}_1,..., \hat{\varrho}_{s'}\}_{\gamma_i \leftrightarrow \gamma_j} \\ (\bigcup_k \bar{\hat{\varrho}}_k) \cap (\bar{\gamma}_i \cup \bar{\gamma}_j) = \varnothing}} \prod_{l=1}^{s'} (\varepsilon\,|z|)^{2|\bar{\hat{\varrho}}_l|/3} \tag{4.34}$$

$$f_1(\gamma_i) = \sum_{\substack{\{\hat{\varrho}_1,..., \hat{\varrho}_{s''}\}_{\gamma_i \leftrightarrow \gamma_i} \\ (\bigcup_k \bar{\hat{\varrho}}_k) \cap \bar{\gamma}_i = \varnothing}} \prod_{l=1}^{s''} (\varepsilon\,|z|)^{2|\bar{\hat{\varrho}}_l|/3} \tag{4.35}$$

In case no such collections of chains appear we put

$$f_2(\gamma_i, \gamma_j) = 1 \tag{4.36}$$

In (4.33) we also used that $0 \in (\bigcup_{l=1}^{s} \hat{\varrho}_l^*) \cup (\bigcup_i \gamma_i^*)$ and $\sum_{l=1}^{s} |\bar{\hat{\varrho}}_l| \geqslant \operatorname{dist}(0, \{\gamma_i\})$.

Lemma 4.5. For every collection of disjoint contours $\{\gamma_1,..., \gamma_r\}$, $r \geqslant 2$ we have the estimate

$$\sum_{\mathscr{G}} \prod_{(\gamma_i, \gamma_j) \in \mathscr{G}} f_2(\gamma_i, \gamma_j) \leqslant 2^{\sum_{i=1}^{r} |\bar{\gamma}_i|} \sum_{\mathscr{T}} \prod_{(\gamma_i, \gamma_j) \in \mathscr{T}} f_2(\gamma_i, \gamma_j) \tag{4.37}$$

Here the sum at the right hand side is taken over all trees $\mathscr{T}$ constructed by using the vertex set $\mathscr{V}$.

Proof. Use Lemma 8 in Chapter 2, Section 4 of ref. 13 to get

$$\sum_{\mathscr{G}} \prod_{(\gamma_i, \gamma_j) \in \mathscr{G}} f_2(\gamma_i, \gamma_j) \leqslant \prod_{(i,j)} (1 + f_2(\gamma_i, \gamma_j)) \sum_{\mathscr{T}} \prod_{(\gamma_i, \gamma_j) \in \mathscr{T}} f_2(\gamma_i, \gamma_j) \tag{4.38}$$

The product $\prod_{(i,j)}$ is taken over all pairs (i, j), $i, j = 1,..., r$.

For every pair (γ_i, γ_j) denote by $\sigma_1^{(i,j)},..., \sigma_m^{(i,j)}$, $m = m(i, j)$ the chains of "free" intervals adjacent to some interval in $\bar{\gamma}_i$ respectively $\bar{\gamma}_j$. We have then

$$1 + f_2(\gamma_i, \gamma_j) = \sum_{\mathscr{S} \subset [1,..., m]} \prod_{l \in \mathscr{S}} (\varepsilon\, |z|)^{2\, |\sigma_l^{(i,j)}|/3} = \prod_{k=1}^{m(i,j)} (1 + (\varepsilon\, |z|)^{2\, |\sigma_k^{(i,j)}|/3}) \leqslant 2^{m(i,j)} \tag{4.39}$$

whenever $\varepsilon\, |z| \leqslant 1$. From here

$$\prod_{j \neq i_0} (1 + f_2(\gamma_{i_0}, \gamma_j)) \leqslant 2^{\sum_{j \neq i_0} m(i_0, j)} \leqslant 2^{2\, |\bar{\gamma}_{i_0}|} \tag{4.40}$$

since the number of chains adjacent to contour γ_0 does not exceed twice the number of intervals in γ_0. In the case (4.36) $1 + f_2(\gamma_i, \gamma_j) = 2$, but estimate (4.40) above stays valid. (4.38) and (4.40) then imply the lemma. ∎

In a similar way we prove also that

$$\prod_{i=1}^{r} f_1(\gamma_i) \leqslant 2^{\sum_i |\bar{\gamma}_i|} \tag{4.41}$$

Denote $\operatorname{dist}(\gamma, \gamma') = \min_{\tau \in \gamma, \tau' \in \gamma'} \operatorname{dist}(\tau, \tau')$, $\operatorname{dist}(\tau_i, \tau_j) = |j-i-1|$, $j \neq i$. For every collection $\{\hat{\varrho}_1, \ldots, \hat{\varrho}_{s'}\}$ entering the sum in $f_2(\gamma_i, \gamma_j)$ we have

$$\sum_{l=1}^{s'} |\bar{\varrho}_l| \geqslant \operatorname{dist}(\gamma_i, \gamma_j) \tag{4.42}$$

From here we have

$$f_2(\gamma_i, \gamma_j) \leqslant (\varepsilon\, |z|)^{\operatorname{dist}(\gamma_i, \gamma_j)/3}\, \tilde{f}_2(\gamma_i, \gamma_j) \tag{4.43}$$

where $\tilde{f}_2$ is defined similarly to f_2 with $(\varepsilon\, |z|)^{2/3}$ replaced by $(\varepsilon\, |z|)^{1/3}$. Repeating the arguments from the proof of Lemma 4.5 and estimates (4.41) and (4.43) we get for any tree $\mathscr{T}$

$$\prod_{(\gamma_i, \gamma_j) \in \mathscr{T}} f_2(\gamma_i, \gamma_j) \prod_{i=1}^{r} f_1(\gamma_i) \leqslant 2^{4 \sum_{i=1}^{r} |\bar{\gamma}_i|} \prod_{(\gamma_i, \gamma_j) \in \mathscr{T}} (\varepsilon\, |z|)^{\operatorname{dist}(\gamma_i, \gamma_j)/3} \tag{4.44}$$

Thus we arrive at the estimate

$$\begin{aligned} &\sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ \Gamma \supset \text{ more than one contour}}} |K_\Gamma|\, |z|^{|\bar{\Gamma}|} \\ &\leqslant \sum_{r=2}^{\infty} \sum_{\{\gamma_1, \ldots, \gamma_r\}} \prod_{i=1}^{r} (16\, |z|\, (1+\varepsilon\, |z|))^{|\bar{\gamma}_i|}\, \mathscr{D}(\gamma_i) (\varepsilon\, |z|)^{\operatorname{dist}(0, \{\gamma_i\})/3} \\ &\quad \times \sum_{\mathscr{T}} \prod_{(\gamma_i, \gamma_j) \in \mathscr{T}} (\varepsilon\, |z|)^{\operatorname{dist}(\gamma_i, \gamma_j)/3} \end{aligned} \tag{4.45}$$

Next we pass to summation over trees. First, we take the trees constructed from the vertex set $\{1, \ldots, r\}$ obtained through $\gamma_k \mapsto k$, $\forall k = 1, \ldots, r$, and denote them by $\tilde{\mathscr{T}}$. Then we resum in (4.45):

$$\begin{aligned} \text{r.h.s. (4.45)} \leqslant &\sum_{r \geqslant 2} \frac{1}{r!} \sum_{\tilde{\mathscr{T}}} \sum_{(\gamma_1, \ldots, \gamma_r)} \prod_{i=1}^{r} (16\, |z|\, (1+\varepsilon\, |z|))^{|\bar{\gamma}_i|} (\varepsilon\, |z|)^{\operatorname{dist}(0, \{\gamma_i\})/3}\, \mathscr{D}(\gamma_i) \\ &\times \prod_{(i,j) \in \tilde{\mathscr{T}}} (\varepsilon\, |z|)^{\operatorname{dist}(\gamma_i, \gamma_j)/3} \end{aligned} \tag{4.46}$$

The third sum here is performed over all ordered collections of contours $\{\gamma_i\}$ such that $\bar{\gamma}_i \cap \bar{\gamma}_j = \varnothing$ whenever $i \neq j$. Since

$$(\varepsilon\, |z|)^{\operatorname{dist}(0, \{\gamma_i\})/3} \leqslant \sum_{i=1}^{r} (\varepsilon\, |z|)^{\operatorname{dist}(0, \gamma_i)/3} \tag{4.47}$$

we get that the right hand side of (4.46) is less than

$$\sum_{r=2}^{\infty} \frac{1}{r!} \sum_{\tilde{\mathscr{T}}} \sum_{i_0=1}^{r} \sum_{\{\gamma_1,\dots,\gamma_r\}} \prod_{i=1}^{r} (16\,|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_i|}\,(\varepsilon\,|z|)^{\mathrm{dist}(0,\,\gamma_{i_0})/3}\,\mathscr{D}(\gamma_i)$$
$$\times \prod_{(i,j)\in\tilde{\mathscr{T}}} (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\,\gamma_j)/3} \tag{4.48}$$

Fix $\tilde{\mathscr{T}}$ and i_0, and estimate first

$$\sum_{\{\gamma_1,\dots,\gamma_r\}} \prod_{i=1}^{r} (16\,|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_i|}\,\mathscr{D}(\gamma_i) \prod_{(i,j)\in\tilde{\mathscr{T}}} (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\,\gamma_j)/3} \tag{4.49}$$

Let $j_0 \neq i_0$ be an extremal vertex of tree $\tilde{\mathscr{T}}$ being joint only with vertex k_0. Then

$$\sum_{\gamma_{j_0}} (16\,|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_0}|}\,\mathscr{D}(\gamma_{j_0})(\varepsilon\,|z|)^{\mathrm{dist}(\gamma_{k_0},\,\gamma_{j_0})/3}$$
$$\leqslant \sum_{\tau''\in\bar{\gamma}_{k_0}} \sum_{\gamma_{j_0}} \sum_{\tau'\in\bar{\gamma}_{j_0}} (\varepsilon\,|z|)^{\mathrm{dist}(\tau',\,\tau'')/3}\,(16\,|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_0}|}\,\mathscr{D}(\gamma_{j_0})$$
$$\leqslant \sum_{\tau''\in\bar{\gamma}_{k_0}} \sum_{\tau'} (\varepsilon\,|z|)^{\mathrm{dist}(\tau',\,\tau'')/3} \sum_{\gamma_{j_0}:\,\tau'\in\bar{\gamma}_{j_0}} (16\,|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_0}|}\,\mathscr{D}(\gamma_{j_0}) \tag{4.50}$$

Here we used the estimate $(\varepsilon\,|z|)^{\mathrm{dist}(\gamma,\,\gamma')/3} \leqslant \sum_{\tau\in\gamma,\,\tau'\in\gamma'} (\varepsilon\,|z|)^{\mathrm{dist}(\tau,\,\tau')/3}$, obtained similarly as in (4.47).

Lemma 4.6. There exists a constant $\bar{C} > 0$ such that for any interval τ and number $k \geqslant 2$

$$\sum_{\substack{\gamma:\,\bar{\gamma}\ni\tau\\ |\bar{\gamma}|=k}} \mathscr{D}(\gamma) \leqslant (\bar{C}\varepsilon)^{k-1} \tag{4.51}$$

Proof.

$$\sum_{\substack{\gamma:\,\tau\in\bar{\gamma}\\ |\bar{\gamma}|=k}} \mathscr{D}(\gamma) = \sum_{\tau_1,\dots,\tau_k} \sum_{\mathscr{G}} \prod_{(\tau_i,\tau_j)\in\mathscr{G}} \frac{\varepsilon}{|i-j-1|^{\delta}+1} \tag{4.52}$$

Here $\mathscr{G} = \mathscr{G}(\tau_1,\dots,\tau_k)$ denotes connected graphs with vertices $\tau_1,\dots,\tau_k$. Note that for fixed i_0

$$\sum_{\tau_j:\, j\neq i_0} \frac{\varepsilon}{|i_0-j-1|^\delta+1} = 2\varepsilon + 2\sum_{k=1}^{\infty} \frac{\varepsilon}{k^\delta+1}$$
$$\leqslant 2\varepsilon + 2\varepsilon \sum_{k=1}^{\infty} \frac{1}{k^\delta}$$
$$\leqslant \bar{C}\varepsilon \tag{4.53}$$

with some $\bar{C} > 0$. Thus by using (4.38) we find

$$\sum_{\mathcal{G}} \prod_{(\tau_i,\tau_j)\in\mathcal{G}} \frac{\varepsilon}{|i-j-1|^\delta+1} \leqslant e^{k\bar{C}\varepsilon} \sum_{\mathcal{T}} \prod_{(\tau_i,\tau_j)\in\mathcal{T}} \frac{\varepsilon}{|i-j-1|^\delta+1} \tag{4.54}$$

Now order the collection $\{\tau_1,..., \tau_k\}$ to get further

$$\sum_{\{\tau_1,...,\tau_k\}} \sum_{\mathcal{T}} = \frac{1}{k!} \sum_{\tilde{\mathcal{T}}} \sum_{(\tau_1,...,\tau_k)} \prod_{(i,j)\in\tilde{\mathcal{T}}} \frac{\varepsilon}{|i-j-1|^\delta+1} \tag{4.55}$$

At the right hand side the summation goes over all trees $\tilde{\mathcal{T}}$ constructed on the vertices $\{1,..., k\}$. By the same argument as before we then obtain

$$\sum_{(\tau_1,...,\tau_k)} \prod_{(\tau_i,\tau_j)\in\mathcal{T}} \frac{\varepsilon}{|i-j-1|^\delta+1} \leqslant (\bar{C}\varepsilon)^{k-1} \tag{4.56}$$

Since the number of trees having k vertices is k^{k-2},[13] by using Stirling's formula, (4.54) and (4.55) we get the result. This completes the proof of the lemma. ▮

From (4.51) we obtain that (4.50) is less than

$$\sum_{\tau''\in\bar{\gamma}_{k_0}} \sum_{\tau'} (\varepsilon\,|z|)^{\mathrm{dist}(\tau',\tau'')/3} \sum_{k=2}^{\infty} (16\,|z|\,(1+\varepsilon\,|z|))^k\,(\bar{C}\varepsilon)^{k-1}$$
$$\leqslant |\bar{\gamma}_{k_0}|\, \frac{16\bar{C}\varepsilon\,|z|\,(1+\varepsilon\,|z|)^2}{(1-(\varepsilon\,|z|)^{1/3})(1-16\bar{C}\varepsilon\,|z|\,(1+\varepsilon\,|z|))} \tag{4.57}$$

From now on we choose z such that $16\bar{C}\varepsilon\,|z|\,(1+\varepsilon\,|z|) < 1$.

Next we go on by taking the next vertex of $\tilde{\mathcal{T}}$, say $j_1 \neq i_0$ connecting with k_1 (that is, we repeat the procedure for the new tree obtained by deleting from $\tilde{\mathcal{T}}$ the vertex j_0 and edge (j_0, k_0). If $j_1 \neq k_0$, we get again an estimate of the type (4.57). If $j_1 = k_0$, then we estimate

$$\sum_{\tau''\in\gamma_{k_1}}\sum_{\tau'}(\varepsilon\,|z|)^{\operatorname{dist}(\tau',\tau'')/3}\sum_{\gamma_{j_1}:\tau'\in\gamma}|\bar{\gamma}_{j_1}|\,(16\,|z|\,(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_1}|}\,\mathscr{D}(\gamma)$$
$$\leqslant\frac{2\,|\bar{\gamma}_{k_1}|}{1-(\varepsilon\,|z|)^{1/3}}\sum_{k=2}^{\infty}(16\,|z|\,(1+\varepsilon\,|z|))^k\,k(\bar{C}\varepsilon)^{k-1}$$

Continuing this procedure inductively we get that after summation over γ_{j_m}, $j_m\neq i_0$, connected with γ_{k_m}, that its net contribution is

$$\frac{2\,|\bar{\gamma}_{k_m}|}{1-(\varepsilon\,|z|)^{1/3}}\sum_{k=1}^{\infty}(16\,|z|\,(1+\varepsilon\,|z|))^k\,k^{l_{j_m}-1}(\bar{C}\varepsilon)^{k-1}$$
$$\leqslant\frac{2\,|\gamma_{k_m}|}{1-\varepsilon\,|z|}\sum_{k=2}^{\infty}(\hat{C}\,|z|)^k\,k^{l_{j_m}-1}(\bar{C}\varepsilon)^{k-1}$$
$$\leqslant\text{const}\,|\gamma_{k_m}|\,\hat{C}\,|z|\sum_{k=2}^{\infty}(\tilde{C}\varepsilon\,|z|)^{k-1}\,k^{l_{j_1}-1}$$

where l_{j_m} is the degree of vertex j_m, i.e., the number of edges of $\tilde{\mathscr{T}}$ incident to j_m, and $\tilde{C}=\bar{C}\hat{C}$.

Lemma 4.7. We have the bound

$$\sum_{k=2}^{\infty}(\tilde{C}\varepsilon\,|z|))^{k-1}\,k^m\leqslant\frac{2^m\tilde{C}e\varepsilon\,|z|\,m!}{1-\tilde{C}e\varepsilon\,|z|}\tag{4.58}$$

Proof. With the notation $x=-\ln(\tilde{C}\varepsilon\,|z|)$ to be used below we have

$$\begin{aligned}\sum_{k=2}^{\infty}(\tilde{C}\varepsilon\,|z|)^{k-1}\,k^m&\leqslant 2^m\sum_{k=1}^{\infty}(\tilde{C}\varepsilon\,|z|)^k\,k^m\\&=2^m\left(-\frac{d}{dx}\right)^m\sum_{k=1}^{\infty}e^{-kx}\\&=2^m\frac{m!}{2\pi i}\int_{|x-\varpi|=1}\frac{e^{-\varpi}}{1-e^{-\varpi}}\,\frac{d\varpi}{(\varpi-x)^{m+1}}\\&\leqslant\frac{2^m\tilde{C}\varepsilon\,|z|\,em!}{1-\tilde{C}e\varepsilon\,|z|}\end{aligned}\tag{4.59}$$

In the first inequality above the bound $k+1\leqslant 2k$ for $k\geqslant 1$, and in the second equality Cauchy's integral formula has been used. ∎

By using Lemma 4.7 above we estimate (4.49) further for fixed γ_{i_0} and $\tilde{\mathscr{T}}$

$$\sum_{\substack{\gamma_k:\, k \neq i_0 \\ k=1,\dots,r}} \prod_{(i,j) \in \tilde{\mathscr{T}}} (\varepsilon\, |z|)^{\operatorname{dist}(\tau_i, \tau_j)/3} \prod_{i \neq i_0} \mathscr{D}(\gamma_i)(16\, |z|\, (1+\varepsilon\, |z|))^{|\bar{\gamma}_i|}$$
$$\leqslant |\bar{\gamma}_{i_0}|^{l_{i_0}} \prod_{k \neq i_0} 2^{l_{jk}} (l_{j_k}-1)!\, (B\varepsilon\, |z|^2)^{r-1}$$

where we used that $\varepsilon\, |z| \leqslant 1/(\tilde{C}e)$ and $B = \text{const}$. Thus we need to estimate (see (4.48))

$$\sum_{\gamma_{i_0}} |\bar{\gamma}_{i_0}|^{l_{i_0}}\, \mathscr{D}(\gamma_{i_0})(16\, |z|\, (1+\varepsilon\, |z|))^{|\gamma_{i_0}|}\, (\varepsilon\, |z|)^{\operatorname{dist}(0, \gamma_{i_0})/3}$$
$$\leqslant \sum_{\tau_k} (\varepsilon\, |z|)^{\operatorname{dist}(0, \tau)} \sum_{\gamma_i:\, \tau \in \gamma_{i_0}} |\bar{\gamma}_{i_0}|^{l_{i_0}}\, \mathscr{D}(\gamma_{i_0})(16\, |z|\, (1+\varepsilon\, |z|))^{|\gamma_{i_0}|}$$

By repeating the arguments above we get

$$\sum_{\gamma_{i_0}:\, \tau \in \gamma_{i_0}} |\bar{\gamma}_{i_0}|^{l_{i_0}}\, \mathscr{D}(\gamma_{i_0})(16\, |z|\, (1+\varepsilon\, |z|))^{|\gamma_{i_0}|} \leqslant \sum_{k=2}^{\infty} k^{l_{i_0}} (\bar{C}\varepsilon)^{k-1}\, (16\, |z|\, (1+\varepsilon\, |z|))^k$$
$$\leqslant \bar{B} 2^{l_{i_0}} l_{i_0}!\, \varepsilon\, |z|^2$$

Summation over τ gives

$$\sum_{\tau} (\varepsilon\, |z|)^{\operatorname{dist}(0, \tau)} \leqslant \text{const} \tag{4.60}$$

hence we finally obtain for fixed $\tilde{\mathscr{T}}$ and i_0

$$2^{l_{i_0}} l_{i_0}! \prod_{j_k \neq i_0} 2^{l_k} (l_{j_k}-1)!\, c(B\varepsilon\, |z|^2)^{r-1} \leqslant c(2^2 B\varepsilon\, |z|^2)^{r-1} \prod_{k=1}^{r} l_{j_k}!$$

where we used the fact that $\sum_{k=1}^{r} l_{j_k} = 2(r-1)$ holding for trees. An upper bound of the number of trees with vertices $\{1,\dots, r\}$ and incidence numbers $\{l_1,\dots, l_r\}$ is[13]

$$\frac{2^{r-2}(r-2)!}{\prod_{j=1}^{r} l_j!} \tag{4.61}$$

Moreover we have that the number of collections $\{l_1,\dots, l_r\}$ such that $l_i > 0$ and $\sum_i l_i = 2(r-1)$ is bounded from above by $2^{2(r-1)}$. By summing over i_0 and putting this estimate together with (4.61), we get

$$\sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ \Gamma \supset \text{more than one contour}}} K_\Gamma\, |z|^{|\bar{\Gamma}|} \leqslant c\varepsilon\, |z|^2 \tag{4.62}$$

with some constant $c > 0$.

Note that the first term appearing in (4.31) can be handled in a similar way than the one leading up to Lemma 4.6, and the same result is obtained. We see then that by choosing z such that $\varepsilon |z|^2 \leqslant$ const, the sum $\sum_{\Gamma} K_\Gamma z^{|\bar{\Gamma}|}$ converges and is bounded. Hence we get that $H(z)$ is an analytic function within a circle of radius $R(\lambda)$ with $R(\lambda) \to \infty$ as $\lambda \to 0$, and is bounded by a constant independent of λ. Thus

$$\sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ |\bar{\Gamma}| = n}} |K_\Gamma| \leqslant \text{const } R(\lambda)^{-n} := \text{const } \eta(\lambda)^n \tag{4.63}$$

with appropriate constants. This completes the proof of Proposition 4.4 and thus of Theorem 2.1. ∎

5. TYPICAL PATH BEHAVIOUR

An important aspect of the problem is to understand what a typical path configuration looks like under the Gibbs measure. This is answered by the following theorem

Theorem 5.1. With μ-probability 1 we have

$$|X_t| \leqslant (C \log(|t|+1))^{1/(s+1)} + Q(X) \tag{5.1}$$

with a suitable number $C > 0$ and a functional $Q(X)$, independent of t.

The strategy of proving Theorem 5.1 is to transfer the typical behaviour for the reference process to the level of the Gibbs measure.

Lemma 5.2. Take an arbitrary $a > 0$. Then there exist $C > 0$ and $\theta > 0$ such that

$$P(\{X \in \mathscr{X} : \max_{0 \leqslant t \leqslant 1} |X_t| \geqslant a\}) \leqslant C e^{-\theta a^{s+1}} \tag{5.2}$$

Proof. For the reference process we have the Dirichlet operator on $L^2(\mathbb{R}^d, dv)$

$$Lf = -\Delta f + 2(\nabla \log \psi_0, \nabla f) \tag{5.3}$$

The corresponding Dirichlet form is

$$\mathscr{E}(f, f) = -\int f \Delta f \, dv + 2 \int f(\nabla \log \psi_0, \nabla f) \, dv \tag{5.4}$$

Now, by using Lemma 1.12 in ref. 10 we have for an arbitrary function $f \in L^2(dv)$ and every $N > 0$

$$P(\{X \in \mathscr{X} : \max_{0 \leqslant t \leqslant 1} |f(X_t)| \geqslant N\}) \leqslant \frac{3}{N} \sqrt{\mathscr{E}(f, f) + (f, f)} \tag{5.5}$$

Choose f in the form of convolution

$$f = f_a := 1_{\{x \in \mathbb{R}^d : |x| \geqslant a\}} * \varphi \tag{5.6}$$

by picking a mollifier φ (with $\|\varphi\|_\infty < \infty$) such that the edges of the indicator function become smooth enough so that the above convolution falls in the domain of L. The mollifier can be chosen so that the smoothing actually takes place around some sphere $S(a)$ of radius a centred at the origin. More precisely, we choose it such that with a suitable $\varepsilon > 0$ we have $f_a(x) = 1$ for $x \in \mathbb{R}^d \setminus S(a+\varepsilon)$, $f_a(x) = 0$ for $x \in S(a-\varepsilon)$, and f_a is some smooth enough function $\tilde{f}_a$ otherwise. Denote these three domains by I, II and III, respectively. Moreover, put $N = 1$ in (5.5); this corresponds to the situation of the path having gone beyond the given level set some time within the given time interval. Thus

$$P(\max_{0 \leqslant t \leqslant 1} |X_t| \geqslant a) \leqslant 3 \sqrt{\|f_a\|^2_{L^2(dv)} + (f_a, Lf_a)_{L^2(dv)}} \tag{5.7}$$

We have

$$\|f_a\|^2_{L^2(dv)} = \int f_a^2(x)\, \psi_0^2(x)\, d\lambda^d = \int_{\mathrm{I}} \psi_0^2(x)\, d\lambda^d + \int_{\mathrm{III}} \tilde{f}_a^2(x)\, \psi_0^2(x)\, d\lambda^d \tag{5.8}$$

By using the bound (4.21) of the ground state, we estimate (and similarly for III)

$$\int_{\mathrm{I}} \psi_0^2(x)\, d\lambda^d \leqslant c e^{-\theta a^{s+1}} \tag{5.9}$$

where $c, \theta > 0$ are independent of a. On the other hand since $\tilde{f}_a$ is smooth enough and $\max_{\mathrm{III}} \{|\nabla \tilde{f}_a|, |\Delta \tilde{f}_a|, \Delta \tilde{f}_a^2\} \leqslant m < \infty$, we get

$$(\tilde{f}_a, L\tilde{f}_a) \leqslant c' e^{-\theta a^{s+1}} \tag{5.10}$$

with suitable $c' > 0$. A similar estimate is obtained also for the other two domains. ∎

Theorem 5.1 will be proven once we will have shown

Lemma 5.3. Suppose there exist some numbers $c, \theta > 0$ such that for any $a > 0$

$$P(\{X \in \mathscr{X} : \max_{0 \leqslant t \leqslant 1} |X_t| \geqslant a\}) \leqslant c\, e^{-\theta a^{s+1}} \tag{5.11}$$

Then there exist $c' > 0$ and $\theta' > 0$ such that for any $a > 0$

$$\mu(\{X \in \mathscr{X} : \max_{0 \leqslant t \leqslant 1} |X_t| \geqslant a\}) \leqslant c' e^{-\theta' a^{s+1}} \tag{5.12}$$

Proof. Note that

$$\mu(\{X \in \mathscr{X} : \max_{0 \leqslant t \leqslant 1} |X_t| \geqslant a\}) = \int \chi_a(X)\, d\mu(X) := \mathscr{M}(a) \tag{5.13}$$

where

$$\chi_a(X) = \begin{cases} 0 & \text{if } \max_{0 \leqslant t \leqslant 1} |X_t| < a \\ 1 & \text{if } \max_{0 \leqslant t \leqslant 1} |X_t| \geqslant a \end{cases} \tag{5.14}$$

On the other hand

$$\mathscr{M}(a) = \lim_{T \to \infty} \frac{\int e^{-\lambda \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, t-s)\, dt\, ds} \chi_a(X)\, d\mathscr{P}(X)}{\int e^{-\lambda \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, t-s)\, dt\, ds}\, d\mathscr{P}(X)} := \lim_{T \to \infty} \frac{Z_T(\chi_a)}{Z_T} \tag{5.15}$$

Here $\mathscr{P}$ is the measure introduced by (3.11). By the cluster expansion we have

$$\begin{aligned} Z_T(\chi_a) &= \mathbb{E}_{\mathscr{P}}[\chi_a] \Bigg(1 + \sum_{r \geqslant 1} \sum_{\substack{\Gamma_1, \ldots, \Gamma_r \\ \Gamma_i^* \cap \Gamma_j^* = \varnothing,\, i \neq j \\ \tau_{N/2} \notin \bigcup_i \bar{\Gamma}_i}} \prod_{i=1}^{r} K_{\Gamma_i}\Bigg) \\ &\quad + \sum_{\Gamma_0 :\, \bar{\Gamma}_0 \ni \tau_{N/2}} \mathbb{E}_{\mathscr{P}}[\kappa_\Gamma^a] \Bigg(1 + \sum_{n \geqslant 1} \sum_{\substack{\Gamma_1, \ldots, \Gamma_r \\ \Gamma_i^* \cap \Gamma_j^* = \varnothing,\, i \neq j \\ \Gamma_0^* \cap (\bigcup_i \Gamma_i^*) = \varnothing}} \prod_{l=1}^{r} K_{\Gamma_l}\Bigg) \\ &= \mathbb{E}_{\mathscr{P}}[\chi_a]\, Z_T^\tau + \sum_{\Gamma :\, \bar{\Gamma} \ni \tau_{N/2}} \mathbb{E}_{\mathscr{P}}[\kappa_\Gamma^a]\, Z_T^\Gamma \end{aligned} \tag{5.16}$$

where $\tau_{N/2} = [0, b]$ and $\kappa_\Gamma^a = \chi_a \kappa_\Gamma$. Here Z_T^τ and Z_T^Γ mean the partition functions over $[-T, T] \backslash \tau_{N/2}$ and $[-T, T] \backslash (\bar{\Gamma} \cup \bar{\Gamma}^\partial)$, respectively. Here we denoted by $\bar{\Gamma}^\partial$ the union of intervals having a common time-point with intervals of $\bar{\Gamma}$. From (5.16)

$$\mathbb{E}_\mu[\chi_a] = \mathbb{E}_{\mathscr{P}}[\chi_a] \frac{Z_T^\tau}{Z_T} + \sum_{\Gamma_0 : \bar{\Gamma}_0 \ni \tau_{N/2}} \mathbb{E}_{\mathscr{P}}[\kappa_{\Gamma_0}^a] \frac{Z_T^{\Gamma_0}}{Z_T} \tag{5.17}$$

follows. By general results in ref. 13 the limits $\lim_{T \to \infty} Z_T^\tau / Z_T$ respectively $\lim_{T \to \infty} Z_T^{\Gamma_0} / Z_T$ exist, and moreover they are estimated by

$$\frac{Z_T^\tau}{Z_T} < 2^3, \qquad \frac{Z_T^{\Gamma_0}}{Z_T} \leqslant 2^{|\bar{\Gamma}_0 \cup \bar{\Gamma}_0^\partial|} \leqslant 2^{3|\bar{\Gamma}_0|} \tag{5.18}$$

The latter estimate comes from the fact that with every interval in $\bar{\Gamma}_0$ at most two intervals from $\bar{\Gamma}_0^\partial$ can be connected. Hence

$$\mathbb{E}_\mu[\chi_a] \leqslant 2^3 \mathbb{E}_{\mathscr{P}}[\chi_a] + \sum_{\Gamma_0 : \bar{\Gamma}_0 \ni \tau_{N/2}} 2^{3|\bar{\Gamma}_0|} \mathbb{E}_{\mathscr{P}}[\kappa_{\Gamma_0}^a] \tag{5.19}$$

On the other hand we have

$$\mathbb{E}_{\mathscr{P}}[\chi_a] = \int \chi_a \frac{dP(X_{\tau_0} \mid q_0, q_1)}{g_b(q_0 \mid q_1)} g_b(q_0 \mid q_1)\, dv(q_0)\, dv(q_1) \leqslant \text{const}\, \mathbb{E}_P[\chi_a] \tag{5.20}$$

where we used that $g_b(q_1 \mid q_0) \geqslant \text{const}$ in the similar way as at Lemma 4.6. Furthermore,

$$\mathbb{E}_{\mathscr{P}}[\kappa_{\Gamma_0}^a] \leqslant (\mathbb{E}_{\mathscr{P}}[\chi_a])^{1/2} (\mathbb{E}_{\mathscr{P}}[\kappa_{\Gamma_0}^2])^{1/2} \tag{5.21}$$

By the same arguments as in Section 4.3 before we obtain the bound

$$\sum_{\Gamma_0 : \bar{\Gamma}_0 \ni \tau_{N/2}} \mathbb{E}_{\mathscr{P}}[\kappa_{\Gamma_0}^2]^{1/2}\, 2^{3|\bar{\Gamma}_0|} \leqslant \text{const} \tag{5.22}$$

with some constant. Hence we get that

$$\mathbb{E}_\mu[\chi_a] \leqslant \text{const}_1\, \mathbb{E}_P[\chi_a] + \text{const}_2\, \mathbb{E}_P[\chi_a]^{1/2} \tag{5.23}$$

Lemma 5.2 then implies (5.12). ▮

We have thus for the stationary measure μ that

$$\mu(\max_{n \leqslant t \leqslant n+1} |X_t| \geqslant (k \log n)^{1/(s+1)}) \leqslant \text{const} \frac{1}{n^{k\theta'}} \tag{5.24}$$

Now choose k so that $k\theta' > 1$. An application of the Borel–Cantelli lemma implies then that μ-almost surely

$$|X_t| \leqslant (k \log t)^{1/(s+1)} \tag{5.25}$$

for $t \geqslant T^*$, with $T^* = T^*(X)$ sufficiently large. Then put $Q(X) = \max_{|t| \leqslant T^*} |X_t|$. This then completes the proof of the theorem. ∎

Remark. It may be conjectured that a result similar to Theorem 5.1 can be obtained from the following fact proven in ref. 1 on the reference measure. The set of paths satisfying

$$\liminf_{|t| \to \infty} |t|^\delta \psi_0(X_t) > 0, \qquad \forall \delta > 1 \tag{5.26}$$

has P-measure 1. Then we expect that a similar property holds also for the Gibbs measure μ.

6. SOME ADDITIONAL PROPERTIES OF GIBBS MEASURES FOR BOUNDED PAIR POTENTIALS

We devote this final section to proving some further properties of the Gibbs measure for (A2-2) type pair potentials. This case in particular covers Nelson's scalar field model which is further discussed in refs. 12 and 2. In the first subsection we look at certain properties of the Gibbs measure like its single-time distributions, conditional distributions and covariances. In the second subsection we establish its uniqueness separately for the cases $\alpha > 2$ with no restriction on the coupling constant, respectively $\alpha > 1$ for sufficiently weak couplings.

6.1. Some Further Properties of the Gibbs Measure

Theorem 6.1. Consider the Gibbs measure μ on $(\mathscr{X}, \mathscr{A})$ for V satisfying assumption (A1) and W satisfying (A2-2). Then the following hold:

1. μ is invariant with respect to time shift and time reflection, i.e.,

$$\mu \circ \tau_t = \mu, \qquad \forall t \in \mathbb{R}, \quad \text{where } (\tau_s X)_t = X_{s+t}$$
$$\mu \circ \vartheta = \mu, \qquad \text{where } (\vartheta X)_t = X_{-t}$$

2. The distributions φ_T of positions X_0 at time $t = 0$ generated by μ_T are absolutely continuous with respect to v, i.e., there exist $C_1, C_2 \in \mathbb{R}$, independent of T and X such that

$$C_1 \leqslant \frac{d\varphi_T}{dv}(q) \leqslant C_2 \tag{6.1}$$

for every $q \in \mathbb{R}^d$ and $T > 0$. Moreover $\lim_{T \to \infty}(d\varphi_T/dv)(q) = (d\varphi/dv)(q)$ exists pointwise.

3. The conditional distributions $\mu_T(\cdot \mid X_0 = q)$ converge locally weakly to $\mu(\cdot \mid X_0 = q)$, for all $q \in \mathbb{R}^d$.

4. For any bounded functions $F^{(1)}, F^{(2)}$ on $\mathbb{R}^d$ we have the estimate on the covariance

$$\mathrm{cov}_\mu(F_s^{(1)}; F_t^{(2)}) = \mathbb{E}_\mu[F_s^{(1)} F_t^{(2)}] - \mathbb{E}_\mu[F_s^{(1)}]\, \mathbb{E}_\mu[F_t^{(2)}]$$
$$|\mathrm{cov}_\mu(F_s^{(1)}; F_t^{(2)})| \leqslant \mathrm{const}\, \frac{\sup |F_s^{(1)}| \sup |F_t^{(2)}|}{|t-s|^\beta + 1}$$

where $\beta > 0$, $F_s^{(1)} := F^{(1)}(X_s)$, $F_t^{(2)} := F^{(2)}(X_t)$, and the constant prefactor is independent of s, t and $F^{(1)}, F^{(2)}$.

Proof. (1) The invariance properties are obvious; they carry over from the reference process and the invariance properties of the pair interaction:

$$W_{[T_1, T_2]}(X) = W_{[T_1+t, T_2+t]}(\tau_t^{-1} X)$$
$$W_{[T_1, T_2]}(X) = W_{[-T_2, -T_1]}(\vartheta X)$$

with arbitrary $T_1 < T_2$, t. Indeed, look at the measures $\mu_{[T_1+t, T_2+t]}$ and $\mu_{[-T_2, -T_1]}$; since the limits $T_1 \to -\infty$, $T_2 \to \infty$ yield the same measure μ, the statements follow.

(2) Notice that

$$\frac{d\varphi_T}{dv}(q) = \frac{Z_T(q)}{Z_T} \tag{6.2}$$

where

$$Z_T(q) = \int e^{-\int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, s-t)\, ds\, dt}\, d\mathscr{P}_T(X \mid X_0 = q) \tag{6.3}$$

By the cluster expansion we obtain

$$\begin{aligned} Z_T(q) = 1 + \sum_{n \geq 1} \sum_{\substack{\Gamma_1,..., \Gamma_n \\ 0 \notin \bigcup_i \Gamma_i^*}} \prod_{i=1}^{n} K_{\Gamma_i} \\ + \sum_{\Gamma_0 :\, 0 \ni \Gamma_0^*} K_{\Gamma_0}(q) \left(1 + \sum_{p \geq 1} \sum_{\substack{\Gamma_1,..., \Gamma_p \\ \Gamma_0^* \cap (\bigcup_i \Gamma_i^*) = \varnothing}} \prod_{i=1}^{p} K_{\Gamma_i} \right) \end{aligned} \tag{6.4}$$

The first term gives the contribution of all clusters disjoint from $t = 0$. In the second we take Γ_0 containing $t = 0$ and sum over all collections of clusters disjoint from Γ_0. The weight for Γ_0 is obtained like in (3.10) by integration with respect to all variables $\{X_{\tau_i}\}$ and $\{q_k\}$ except for $q_0 = q$. Evidently, for all $q \in \mathbb{R}$, estimate (4.1) for $K_{\Gamma_0}(q)$ stays true. Next we use the general representation for the logarithm of the partition function (see ref. 13):

$$\log Z_T = \sum_{n \geq 1} \sum_{\eta_n := \{\Gamma_1,..., \Gamma_n\}} \mathscr{D}_{\eta_n} \prod_{\Gamma \in \eta_n} K_\Gamma \tag{6.5}$$

The second summation is performed over all connected collections of n clusters. The coefficients $\mathscr{D}_\eta$ are such that (6.5) is absolutely convergent whenever the clusters satisfy estimate (4.1). Moreover, for any fixed $m \in [-N, N]$

$$\sum_{n \geq 1} \sum_{\substack{\eta_n \\ m \in \eta_n^*}} |\mathscr{D}_{\eta_n}| \prod_{\Gamma \in \eta_n} |K_\Gamma| \leq C \tag{6.6}$$

holds, with some number $C > 0$. (We denoted $\eta^* = \bigcup_{\Gamma \in \eta} \Gamma^*$.) Let η^0 denote the collection of all possible Γ_0 containing the origin; the corresponding weights are K_{Γ_0}. Hence we obtain from (6.5)

$$\begin{aligned} &\log Z_T(q) - \log Z_T \\ &= \sum_{n \geq 1} \sum_{\substack{\eta_n :\, \eta_n \cap \eta^0 \neq \varnothing \\ \eta_n^* \subset [-N, N]}} \mathscr{D}_{\eta_n} \prod_{\Gamma \in \eta_n} \hat{K}_\Gamma - \sum_{n \geq 1} \sum_{\substack{\eta_n :\, \eta_n \cap \eta^0 \neq \varnothing \\ \eta_n^* \subset [-N, N]}} \mathscr{D}_{\eta_n} \prod_{\Gamma \in \eta_n} K_\Gamma \end{aligned} \tag{6.7}$$

where

$$\hat{K}_\Gamma = \begin{cases} K_\Gamma & \text{if} \quad \Gamma^* \not\ni 0 \;\; (\Gamma \notin \eta^0) \\ K_\Gamma(q) & \text{if} \quad \Gamma^* \ni 0 \;\; (\Gamma \in \eta^0) \end{cases} \tag{6.8}$$

Clearly,

$$\sum_{n\geqslant 1} \sum_{\substack{\eta_n:\, \eta_n \cap \eta^0 \neq \varnothing \\ \eta_n^* \subset [-N,N]}} \mathscr{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma = \sum_{n\geqslant 1} \sum_{\substack{\eta_n:\, \eta_n^* \ni 0 \\ \eta_n^* \subset [-N,N]}} \mathscr{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma \tag{6.9}$$

and by (6.6) we get

$$\left|\sum_{n\geqslant 1} \sum_{\substack{\eta_n:\, \eta_n \cap \eta^0 \neq \varnothing \\ \eta^* \subset [-N,N]}} \mathscr{D}_{\eta_n}\right| \prod_{\Gamma\in\eta_n} |\hat{K}_\Gamma| \leqslant C \tag{6.10}$$

A similar estimate is obtained for the second term appearing in (6.7). Hence (6.1) follows.

The existence of the limit claimed in (2) follows by (6.7) and

$$\sum_{n\geqslant 1} \sum_{\substack{\eta_n:\, \eta_n \cap \eta^0 \neq \varnothing \\ \eta_n^* \subset [-N,N]}} \mathscr{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma \to \sum_{n\geqslant 1} \sum_{\eta_n:\, \eta_n \cap \eta^0 \neq \varnothing} \mathscr{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma \quad \text{as } N\to\infty \tag{6.11}$$

The convergence above comes about by combining (6.6) with (6.9). The second term in (6.7) can be treated completely similarly.

(3) The convergence of $\mu_T(\cdot \mid X_0 = q)$ to $\mu(\cdot \mid X_0 = q)$ is proved in a similar way as the convergence of μ_T to μ was proven above. In this case K_Γ has to be changed for $\hat{K}_\Gamma$, and as established before, $\hat{K}_{\Gamma_0}(q_0)$ satisfies estimate (4.1).

(4) First note that for any bounded function F on $\mathbb{R}^d$

$$\mathbb{E}_\mu[F_0] = \mathbb{E}_\nu[F_0]\, f_{\{0\}} + \sum_{\Gamma_0:\, 0\ni\Gamma_0^*} K_{\Gamma_0}(F_0)\, f_{\{\Gamma_0^*\}} \tag{6.12}$$

and

$$K_{\Gamma_0}(F_0) = \mathbb{E}_{\mathscr{P}}[F_0 \kappa_{\Gamma_0}] \tag{6.13}$$

where $F_0 = F(X_0)$, see (3.10) and (3.12). Furthermore, $f_A = \lim_{T\to\infty} Z_T^A / Z_T$, and is estimated as before like $|f_A| \leqslant 2^{|A|}$. For $A_1 \cap A_2 = \varnothing$ we moreover have

$$|f_{A_1\cup A_2} - f_{A_1} f_{A_2}| \leqslant \text{const}\, \frac{2^{|A_1|+|A_2|}}{\text{dist}(A_1, A_2)^\zeta} \tag{6.14}$$

with some $\zeta > 0$. This estimate can easily be obtained by the general results from ref. 13. Now we can write

$$\begin{aligned}
\mathbb{E}_\mu[F_s^{(1)} F_t^{(2)}] &= \mathbb{E}_\nu[F_s^{(1)}]\, \mathbb{E}_\nu[F_t^{(2)}]\, f_{\{s,t\}} \\
&\quad + \sum_{\substack{\Gamma_1 \\ \Gamma_1^* \ni s,\, \Gamma_1^* \not\ni t}} \mathbb{E}_{\mathscr{P}}[F_s^{(1)} \kappa_{\Gamma_1}]\, \mathbb{E}_\nu[F_t^{(2)}]\, f_{\{t \cup \Gamma_1^*\}} \\
&\quad + \sum_{\substack{\Gamma_2 \\ \Gamma_2^* \ni t,\, \Gamma_2^* \not\ni s}} \mathbb{E}_\nu[F_s^{(1)}]\, \mathbb{E}_{\mathscr{P}}[F_t^{(2)} \kappa_{\Gamma_2}]\, f_{\{s \cup \Gamma_2^*\}} \\
&\quad + \sum_{\substack{\Gamma_1, \Gamma_2 :\, \Gamma_1^* \cap \Gamma_2^* = \varnothing \\ \Gamma_1 :\, \Gamma_1^* \ni s,\, \Gamma_2 :\, \Gamma_2^* \ni t}} \mathbb{E}_{\mathscr{P}}[F_s^{(1)} \kappa_{\Gamma_1}]\, \mathbb{E}_{\mathscr{P}}[F_t^{(2)} \kappa_{\Gamma_2}]\, f_{\{\Gamma_1^* \cup \Gamma_2^*\}} \\
&\quad + \sum_{\Gamma :\, \Gamma^* \ni s, t} \mathbb{E}_{\mathscr{P}}[F_s^{(1)} F_t^{(2)} \kappa_\Gamma]\, f_{\{\Gamma^*\}}
\end{aligned}$$

From this formula and (6.12) we obtain

$$\begin{aligned}
&\operatorname{cov}_\mu(F_s^{(1)}; F_t^{(2)}) \\
&\quad = \mathbb{E}_\nu[F_s^{(1)}]\, \mathbb{E}_\nu[F_t^{(2)}](f_{\{s,t\}} - f_{\{s\}} f_{\{t\}}) \\
&\qquad + \sum_{\substack{\Gamma_1 \\ \Gamma_1^* \ni s,\, \Gamma_1^* \not\ni t}} \mathbb{E}_\nu[F_t^{(2)}]\, \mathbb{E}_{\mathscr{P}}[F_s^{(1)} \kappa_{\Gamma_1}](f_{\{t \cup \Gamma_1^*\}} - f_{\{t\}} f_{\{\Gamma_1^*\}}) \\
&\qquad + \sum_{\substack{\Gamma_2 \\ \Gamma_2^* \ni t,\, \Gamma_2^* \not\ni s}} \mathbb{E}_\nu[F_s^{(1)}]\, \mathbb{E}_{\mathscr{P}}[F_t^{(2)} \kappa_{\Gamma_2}](f_{\{s \cup \Gamma_2^*\}} - f_{\{s\}} f_{\{\Gamma_2^*\}}) \\
&\qquad + \sum_{\substack{\Gamma_1, \Gamma_2 :\, \Gamma_1^* \cap \Gamma_2^* = \varnothing \\ \Gamma_1^* \ni s,\, \Gamma_2^* \ni t}} \mathbb{E}_{\mathscr{P}}[F_s^{(1)} \kappa_{\Gamma_1}]\, \mathbb{E}_{\mathscr{P}}[F_t^{(2)} \kappa_{\Gamma_2}](f_{\{\Gamma_1^* \cup \Gamma_2^*\}} - f_{\{\Gamma_1^*\}} f_{\{\Gamma_2^*\}}) \\
&\qquad + \sum_{\Gamma :\, \Gamma^* \ni s, t} \mathbb{E}_{\mathscr{P}}[F_s^{(1)} F_t^{(2)} \kappa_\Gamma]\, f_{\{\Gamma^*\}} \\
&\qquad - \sum_{\Gamma_2 :\, \Gamma_2^* \ni s, t} \mathbb{E}_\nu[F_s^{(1)}]\, \mathbb{E}_{\mathscr{P}}[F_t^{(2)} \kappa_{\Gamma_2}]\, f_{\{s\}} f_{\{\Gamma_2^*\}} \\
&\qquad - \sum_{\Gamma_1 :\, \Gamma_1^* \ni s, t} \mathbb{E}_\nu[F_t^{(2)}]\, \mathbb{E}_{\mathscr{P}}[F_s^{(1)} \kappa_{\Gamma_1}]\, f_{\{t\}} f_{\{\Gamma_1^*\}} \\
&\qquad - \sum_{\substack{\Gamma_1, \Gamma_2 :\, \Gamma_1^* \cap \Gamma_2^* \neq \varnothing \\ \Gamma_1^* \ni s,\, \Gamma_2^* \ni t}} \mathbb{E}_{\mathscr{P}}[F_s^{(1)} \kappa_{\Gamma_1}]\, \mathbb{E}_{\mathscr{P}}[F^{(2)} \kappa_{\Gamma_2}]\, f_{\{\Gamma_1^*\}} f_{\{\Gamma_2^*\}}.
\end{aligned}$$

For estimating the first four terms at the r.h.s. of the above sum we use (6.14) along with the bound

$$|\mathbb{E}_{\mathscr{P}}[F^{(i)}\kappa_{\Gamma_i}]| \leqslant \frac{\sup |F^{(i)}|}{(\operatorname{diam} \Gamma^*)^{\zeta'}+1} E_{\Gamma_i}(\delta', \lambda') \tag{6.15}$$

$i=1, 2$, where $E_{\Gamma_i}(\delta', \lambda')$ is the function appearing at the r.h.s. of estimate (4.1) with slightly modified entries (δ', λ' instead of δ, λ; $\delta' > 1$) so that (4.29) still holds. Here $\zeta' = \delta - \delta' > 0$, and we used in addition that

$$\frac{1}{(\operatorname{diam} \Gamma^*)^{\zeta'}+1} \frac{1}{\operatorname{dist}(s, \Gamma^*)^{\zeta}+1} \leqslant \frac{1}{|s-t|^{\beta}+1} \tag{6.16}$$

whenever $t \in \Gamma^*$ (and similarly for $s \in \Gamma^*$), and

$$\frac{1}{(\operatorname{diam} \Gamma_1^*)^{\zeta'}+1} \frac{1}{(\operatorname{diam} \Gamma_2^*)^{\zeta'}+1} \frac{1}{\operatorname{dist}(\Gamma_1^*, \Gamma_2^*)^{\zeta}} \leqslant \frac{1}{|s-t|^{\beta}+1} \tag{6.17}$$

for $s \in \Gamma_1^*$, $t \in \Gamma_2^*$, and $\beta = \min\{\zeta, \zeta'\} > 0$.

Next, in the fifth term above we used that $\operatorname{diam} \Gamma^* \geqslant |t-s|$ when $s, t \in \Gamma^*$, and that

$$|\mathbb{E}_{\mathscr{P}}[F_s^{(1)} F_t^{(2)} \kappa_{\Gamma}]| \leqslant \sup |F_s^{(1)}| \sup |F_t^{(2)}| \frac{E_{\Gamma}(\delta', \lambda')}{(\operatorname{diam} \Gamma^*)^{\zeta'}+1}. \tag{6.18}$$

For the remaining three terms in the sum above we apply the same argument. Thus for the full sum the corresponding bounds become

$$\text{const} \frac{\sup |F_s^{(1)}| \sup |F_t^{(2)}|}{|s-t|^{\beta}+1} E_{\Gamma_1}(\delta', \lambda') E_{\Gamma_2}(\delta', \lambda') 2^{|\Gamma_1^*|+|\Gamma_2^*|} \tag{6.19}$$

whenever $s \in \Gamma_1^*$, $t \in \Gamma_2^*$, respectively

$$\text{const} \frac{\sup |F_1| \sup |F_2|}{|s-t|^{\beta}+1} E_{\Gamma}(\delta', \lambda') 2^{|\Gamma^*|} \tag{6.20}$$

whenever $s \in \Gamma_1^*$ or $t \in \Gamma_2^*$ or $s, t \in \Gamma^*$. Then by summing over Γ_1, Γ_2 respectively Γ, we finally obtain the similar results as in Section 4.3. ▮

6.2. Uniqueness of the Gibbs Measure

In this section we prove uniqueness of the Gibbs measure for two cases. One is for pair potentials of (A2-2) type with exponent $\alpha > 2$ in which case we apply a general argument involving no restriction on the coupling constant. The other is for the same class of pair potentials with exponent $\alpha > 1$ in which case we use cluster expansion and need to restrict to sufficiently small values $|\lambda|$.

First we deal with the $\alpha > 2$ case. Consider $\mu_T(\cdot \mid Y)$ given by

$$d\mu_T(X \mid Y) = \frac{1}{Z_T(Y)} e^{-\lambda W_T(X \mid Y)} d\mathscr{P}_T(X \mid Y) \tag{6.21}$$

Here

$$W_T(X \mid Y) = W_T(X) + W_T^Y(X) \tag{6.22}$$

is the interaction energy in configuration $X \in \mathscr{X}_{[-T,T]}$ given the boundary configuration $Y = Y^- \cup Y^+$, with $Y^- \in \mathscr{X}_{(-\infty,-T]}$ resp. $Y^+ \in \mathscr{X}_{[T,\infty)}$. W_T is as given by the integral in (3.1), and

$$W_T^Y(X) = 2\int_{-\infty}^{-T} dt \int_{-T}^{T} ds\, W(Y_t^-, X_s, t-s) + 2\int_{T}^{\infty} dt \int_{-T}^{T} ds\, W(Y_t^+, X_s, t-s) \tag{6.23}$$

$\mathscr{P}_T(\cdot \mid Y) = \mathscr{P}_T(\cdot \mid Y_{-T}^-, Y_T^+)$ is the conditional distribution of the reference measure for the given boundary condition Y which depends only on the positions attained at $\pm T$ since $\mathscr{P}$ is Markovian. $Z_T(Y)$ is the partition function for $\mu_T(\cdot \mid Y)$. It is easily checked that $\{\mu_T(\cdot \mid Y)\}$, with $Y \in C(\mathbb{R} \setminus [-T, T], \mathbb{R}^d)$, $0 < T < \infty$, is a compatible and regular family of conditional probability measures defining a specification in the sense of DLR theory. The limiting Gibbs measure μ on $\mathscr{X}$ is well defined also in DLR sense and it is consistent with this specification (see ref. 7 for terminology and details).

Theorem 6.2. Suppose V is of (A1) class and W is of (A2-2) class with $\alpha > 2$. Then whenever the Gibbs measure μ exists, it is unique in DLR sense.

Proof. We show that there exists a number $L > 0$ such that for any cylinder function f on $C(\mathbb{R}, \mathbb{R}^d)$ and any pair of Gibbs measures μ_1 and μ_2 consistent with (6.21)

$$\mathbb{E}_{\mu_1}[f] \leqslant L\, \mathbb{E}_{\mu_2}[f] \tag{6.24}$$

First we show that once having (6.24), uniqueness follows. Indeed, suppose that the set of Gibbs measures consistent with (6.21) contains at least two elements. Then, since the set of Gibbs measures consistent with a given specification is convex, by the Krein–Milman theorem it would then follow that at least two extremal Gibbs measures $\mu_1 \neq \mu_2$ existed. Then some event $E \in \mathscr{A}$ would separate them, i.e., we would have $\mu_1(E) = 1$ and $\mu_2(E) = 0$. Equivalently, for any $\varepsilon > 0$ a cylinder event E' would exist such that $\mu_1(E') > 1 - \varepsilon$ and $\mu_2(E') < \varepsilon$. Now choose f above to be the indicator function of this event. Then putting $\varepsilon < 1/(L+1)$ would contradict (6.24).

We now turn to proving (6.24). By using estimate (2.2) with $\alpha > 2$ it is easy to obtain that

$$|W_T^Y(X)| \leqslant c_1 \tag{6.25}$$

uniformly in T and in configurations X and Y, with some constant $c_1 > 0$. Now take $T' = T - \tau$, with some $\tau > 0$ to be specified later on. We take an arbitrary path $X \in \mathscr{X}$ and break it up into $\bar{X}_\tau = X|_{[-T', T']}$, $X_\tau^- = X|_{(-T, -T')}$ and $X_\tau^+ = X|_{[T', T]}$. We have then

$$W_T(X) = W_{T'}(\bar{X}_\tau) + W_{T'}^{X_\tau^- \cup X_\tau^+}(\bar{X}_\tau) \tag{6.26}$$

where in the second term integration is made over $[-T, T]^2 \backslash [-T', T']^2 \subset \mathbb{R}^2$ using the pieces X_τ^-, X_τ^+ of path X. Again, by estimate (2.2) we get

$$|W_{T'}^{X_\tau^- \cup X_\tau^+}(\bar{X}_\tau)| \leqslant c_2 \tau \tag{6.27}$$

uniformly in the pieces of X, with some $c_2 > 0$.

Lemma 6.3. There is a number $M = M(\tau) > 0$ such that for any Y and $T > 0$

$$\frac{1}{M} \leqslant \frac{Z_T(Y)}{Z_{T'}} \leqslant M \tag{6.28}$$

where $Z_{T'}$ is defined by (2.8).

Proof. By (6.25) and (6.27) we obtain

$$\begin{aligned} Z_T(Y) &= \int e^{-\lambda W_T(X \mid Y)}\, d\mathscr{P}_T(X \mid Y) \\ &\leqslant e^{|\lambda|\, c_1} \int e^{-\lambda W_T(X)}\, d\mathscr{P}_T(X \mid Y) \\ &\leqslant e^{|\lambda|\, c_1 + |\lambda|\, c_2 \tau} \int e^{-\lambda W_{T'}(\bar{X}_\tau)}\, d\mathscr{P}_T(X \mid Y) \end{aligned} \tag{6.29}$$

Put for a shorthand $q^{\pm} = (\bar{X}_\tau)_{\pm T'}$, i.e., the positions of $\bar{X}_\tau$ at the endpoints of the interval $[-T', T']$. Then by using the Markov property of the reference measure we have furthermore

$$\int e^{-\lambda W_{T'}(\bar{X}_\tau)}\, d\mathscr{P}_T(X \mid Y)$$

$$= \int e^{-\lambda W_{T'}(\bar{X}_\tau)}\, d\mathscr{P}_{T'}(\bar{X}_\tau \mid q^-, q^+)\, g_\tau(q^- \mid Y^-_{-T})\, g_\tau(q^+ \mid Y^+_T)\, d\nu(q^-)\, d\nu(q^+) \tag{6.30}$$

with g_τ as under (2.6). Using now (4.11) we get that

$$\tfrac{1}{2} \leqslant 1 - Ce^{-A\tau} \leqslant g_\tau(q^{\pm} \mid Y^{\pm}_{\pm T}) \leqslant 1 + Ce^{-A\tau} \leqslant 2 \tag{6.31}$$

for large enough τ. Fixing now such a τ we obtain by (6.29), (6.30) and (6.31) that

$$Z_T(Y) \leqslant 2e^{|\lambda|\,(c_1 + c_2\tau)} Z_{T'} \tag{6.32}$$

Similarly, we get the converse inequality

$$Z_T(Y) \geqslant \tfrac{1}{2} e^{-|\lambda|\,(c_1 + c_2\tau)} Z_{T'} \tag{6.33}$$

(6.32) and (6.33) taken together imply then (6.28) with $M = 2e^{|\lambda|\,(c_1 + c_2\tau)}$. ∎

Now we return to showing (6.24). Let $f = f_{T'}$ be the cylinder function $f_{T'} = f_{T'}(X|_{[-T', T']})$. By using the DLR equation we obtain

$$\mathbb{E}_{\mu_1}[f_{T'}] = \int \frac{f_{T'}(\bar{X}_\tau)\, e^{-\lambda W_T(X \mid Y)}}{Z_T(Y)}\, d\mathscr{P}_T(X \mid Y)\, d\mu_1(Y)$$

$$\leqslant M \frac{1}{Z_{T'}} \int f_{T'}(\bar{X}_\tau)\, e^{-\lambda W_T(X \mid Y)}\, d\mathscr{P}_T(X \mid Y)\, d\mu_1(Y)$$

By the same arguments as above we obtain furthermore

$$\int f_{T'}(\bar{X}_\tau)\, e^{-\lambda W_T(X \mid Y)}\, d\mathscr{P}_T(X \mid Y)\, d\mu_1(Y) \leqslant M \int f_{T'}(\bar{X}_\tau)\, e^{-\lambda W_{T'}(\bar{X}_\tau)}\, d\mathscr{P}_{T'}(\bar{X}_\tau) \tag{6.34}$$

and from here

$$\mathbb{E}_{\mu_1}[f_{T'}] \leqslant M^2 \mathbb{E}_{\mu_{T'}}[f_{T'}] \tag{6.35}$$

This completes the proof of the theorem. ∎

Next we turn to the $\alpha > 1$ case.

Theorem 6.4. Suppose V is of class (A1) and W is of class (A2-2) with $\alpha > 1$. Then for sufficiently small $|\lambda| \neq 0$ the limiting Gibbs measure μ is unique.

Proof. We consider the class of bounded local functions F_S on $C(\mathbb{R}, \mathbb{R}^d)$ indexed by finite intervals $S \subset \mathbb{R}$ (that is, F_S is measurable with respect to $\mathscr{A}_S$). For the uniqueness of the Gibbs measure it suffices to prove that for any increasing sequence $\{T_n\}$, $T_n \to \infty$, and any corresponding sequence of boundary conditions $\{Y_n\}$

$$\lim_{n \to \infty} \mathbb{E}_{\mu_{T_n}(\cdot \mid Y_n)}[F_S] = \mathbb{E}_\mu[F_S] \tag{6.36}$$

for arbitrary F_S of the above class. In order to show this we express the conditional expectations appearing above in terms of the cluster representation. We suppose without loss of generality that S consists of a finite union of intervals of the partition of $[-T, T]$.

From now on we follow the steps of the cluster expansion we explained in Sections 3 and 4 before. Take the same partition of the interval $[-T, T]$ into disjoint segments as before. The interaction energy can then be written as

$$W_T(X \mid Y) = \sum_{0 \leqslant i < j \leqslant N} W_{\tau_i, \tau_j}(X_{\tau_i}, X_{\tau_j}) + \sum_{0 \leqslant i \leqslant N-1} W^Y_{\tau_i, T}(X_{\tau_i}) \tag{6.37}$$

with the same notations as before, and with

$$W^Y_{\tau_i, T}(X_{\tau_i}) = 2 \int_{-\infty}^{T} ds \int_{\tau_i} dt\, W(X_t, Y^-_s, s-t) + 2 \int_{T}^{\infty} ds \int_{\tau_i} dt\, W(X_t, Y^+_s, s-t) \tag{6.38}$$

By (2.2) the estimate

$$|W^Y_{\tau_i, T}| \leqslant \frac{Cb}{(\operatorname{dist}(\tau_i, [-T, T]^c) + 1)^{\alpha - 1}} \tag{6.39}$$

easily follows. (Here $[-T, T]^c = \mathbb{R} \setminus [-T, T]$ and $\operatorname{dist}(\tau_k, [-T, T]^c) = \min\{kb, (N-k-1)\, b\}$.) We fix the positions $X_{t_0} = Y^-_{-T} = q_0$, $X_{t_1} = q_1, \ldots,$ $X_{t_{N-1}} = q_{N-1}$, $X_{t_N} = Y^+_T = q_N$. Similarly to (3.11) we introduce the auxiliary measure

$$d\mathscr{P}^Y_T = \prod_{k=0}^{N-1} \frac{e^{-W^Y_{\tau_k, T}(X_k)}}{\mathscr{Z}^T_{\tau_k}(Y \mid q_k, q_{k+1})}\, dP_b(X_{\tau_k} \mid q_k, q_{k+1}) \prod_{k=1}^{N-1} d\nu_k(q_k) \tag{6.40}$$

where

$$\mathscr{Z}^T_{\tau_k}(Y \mid q_k, q_{k+1}) = \mathbb{E}_{P_b(\cdot \mid q_k, q_{k+1})}[e^{-W^Y_{\tau_k, T}(X_{\tau_k})}] \tag{6.41}$$

Also, for every cluster Γ we consider the function κ^Y_Γ defined similarly to (3.10) by changing the measure $\mathscr{P}_T$ for $\mathscr{P}^Y_T$. If $\pm T \notin \Gamma^*$ then κ^Y_Γ does not depend on Y. If $-T \in \Gamma^*$ and/or $T \in \Gamma^*$ then κ^Y_Γ depends on $Y^-_{-T} = q_0$ and/or $Y^+_T = q_N$, respectively. Next we define the weights like under (3.12):

$$K^Y_\Gamma = \mathbb{E}_{\mathscr{P}^Y_T}[\kappa^Y_\Gamma] \tag{6.42}$$

The partition function $Z_T(Y)$ can be expressed then similarly to (3.13) with these adapted objects. Since Lemma 4.3 holds unchanged and gives a uniform estimate in paths, κ^Y_Γ is estimated in the same way as κ_Γ. Combining thus (6.39) with (4.27) we obtain

$$\begin{aligned} \exp\left(-\frac{C\,|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k, [-T, T]^c)+1)^{\alpha-1}}\right) &\leqslant \frac{\exp(-W^Y_{\tau_k, T}(X_{\tau_k}))}{\mathscr{Z}^Y_{\tau_k}(Y \mid q_k, q_{k+1})} \\ &\leqslant \exp\left(\frac{C\,|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k, [-T, T]^c)+1)^{\alpha-1}}\right) \end{aligned} \tag{6.43}$$

This implies then that

$$\begin{aligned} &|K_\Gamma| \prod_{\substack{\tau_k \in \bigcup \bar{\gamma} \\ \gamma \in \Gamma}} \exp\left(-\frac{C\,|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k, [-T, T]^c)+1)^{\alpha-1}}\right) \\ &\leqslant |K^Y_\Gamma| \leqslant |K_\Gamma| \prod_{\substack{\tau_k \in \bigcup \bar{\gamma} \\ \gamma \in \Gamma}} \exp\left(\frac{C\,|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k, [-T, T]^c)+1)^{\alpha-1}}\right) \end{aligned} \tag{6.44}$$

At this point it is useful to make two remarks:

1. For sufficiently small $|\lambda| \neq 0$ the bound

$$|K^Y_\Gamma| \leqslant 2^{|\bar{\Gamma}|}\,|K_\Gamma| \tag{6.45}$$

holds (the factor 2 can actually be replaced by any number larger than 1). Thus the cluster estimate (4.29) obtained in Proposition 4.4 stays essentially valid:

$$\sum_{\substack{\Gamma:\, \Gamma^* \ni 0 \\ |\bar{\Gamma}| = n}} |K^Y_\Gamma| \leqslant c\eta'(\lambda) \tag{6.46}$$

with $\eta'(\lambda) = 2\eta(\lambda)$, going to zero as $\lambda \to 0$.

2. For any fixed Γ we have

$$\lim_{T \to \infty} \kappa_\Gamma^Y = \kappa_\Gamma \tag{6.47}$$

as noted before, and thus

$$\lim_{T \to \infty} K_\Gamma^Y = K_\Gamma \tag{6.48}$$

both converging uniformly in Y.

Then in (6.36) we have

$$\mathbb{E}_{\mu_T}[F_S|Y] = \mathbb{E}_{\mathscr{P}_T^Y}[F_S] \frac{Z_T^S(Y)}{Z_T(Y)} + \sum_{n \geqslant 1} \sum_{\substack{\{\Gamma_1, ..., \Gamma_m\}: \Gamma_i^* \cap \Gamma_j^* \neq \varnothing \\ \Gamma_i^* \cap S \neq 0, \Gamma_i \subset [-T, T], i=1,...,m}} \mathbb{E}_{\mathscr{P}_T^Y}\left[F_S \prod_{i=1}^m \kappa_{\Gamma_i}^Y \right] \frac{Z_T^{\cup \bar{\Gamma}}(Y)}{Z_T(Y)} \tag{6.49}$$

with the same notations as in (5.16), and specifically $Z_T^{\cup \bar{\Gamma}} = Z_{[-T,T] \setminus \bigcup_{i=1}^m \bar{\Gamma}_i}$.

Take now a collection of intervals $\{\tau_i\} = \mathscr{U}$; the partition function $Z_{[-T,T] \setminus \bigcup_{\tau_i \in \mathscr{U}} \tau_i}(Y) := Z_T^{\mathscr{U}}(Y)$ can then be written like in (3.13) only by changing K_Γ for K_Γ^Y and summing over collections of clusters lying inside $[-T,T] \setminus \bigcup_{\tau_i \in \mathscr{U}} \tau_i$ (i.e., $\Gamma^* \cap \partial \mathscr{U} = \varnothing$).

Lemma 6.5. For sufficiently small $|\lambda| \neq 0$ we have the following properties of $f_T^{\mathscr{U}}(Y) := Z_T^{\mathscr{U}}(Y)/Z_T(Y)$. On the one hand,

$$|f_T^{\mathscr{U}}(Y)| \leqslant 2^{|\mathscr{U}|} \tag{6.50}$$

$|\mathscr{U}|$ denoting the number of intervals contained in $\mathscr{U}$. On the other hand,

$$\lim_{T \to \infty} f_T^{\mathscr{U}}(Y) = f^{\mathscr{U}} \tag{6.51}$$

uniformly in Y. Moreover, $f^{\mathscr{U}}$ also satisfies (6.50) above.

Proof. The estimate (6.50) follows by the same token as explained around (5.18). To obtain (6.51) we use a similar decomposition as in (6.5) to write

$$f_T^{\mathscr{U}}(Y) = 1 + \sum_{\substack{\eta = \{\Gamma\} \\ \Gamma \subset [-T,T]}} \widetilde{\mathscr{D}}_{\mathscr{U}}(\eta) \prod_{\Gamma \in \eta} K_\Gamma^Y \tag{6.52}$$

and

$$f^{\mathscr{U}} = 1 + \sum_{\eta = \{\Gamma\}} \widetilde{\mathscr{D}}_{\mathscr{U}}(\eta) \prod_{\Gamma \in \eta} K_\Gamma \tag{6.53}$$

where both summations go over non-empty collections of clusters $\eta = \{\Gamma\}$ for which $\bigcup_{\Gamma \in \eta} \Gamma^* \cap \mathscr{U}^* \neq \varnothing$ ($\mathscr{U}^*$ is the set of all time points occurring in $\mathscr{U}$), and moreover in the first term the clusters are required to lie inside $[-T, T]$. The coefficients appearing in both sum are such that the series (6.53) is absolutely convergent whenever for the K_Γ occurring there (4.29) is satisfied with some small enough bound $\eta(\lambda)$. Hence by (6.45) we see that the series (6.52) is term by term dominated by $\sum_\eta |\widetilde{\mathscr{D}}(\eta)| \prod_{\Gamma \in \eta} 2^{|\bar{\Gamma}|} |K_\Gamma|$ (here summation goes over all collections of clusters crossing $\mathscr{U}$). Then by putting

$$\hat{K}_\Gamma^Y = \begin{cases} K_\Gamma^Y & \text{if } \Gamma \text{ lies inside } [-T, T] \\ 0 & \text{otherwise} \end{cases} \tag{6.54}$$

and using Lebesgue's dominated convergence theorem we obtain that $f_{\mathscr{U}}^Y \to f_{\mathscr{U}}$ uniformly as $T \to \infty$. ▮

We now return to the expression (6.49). By the ergodicity of the reference measure

$$\lim_{T \to \infty} \mathbb{E}_{P_T^Y}[F_S] = \mathbb{E}_P[F_S] \tag{6.55}$$

and hence the first term of (6.49) converges to $\mathbb{E}_{\mathscr{P}}[F_S]\, f_S$. By the same argument as above we obtain also

$$\lim_{T \to \infty} \mathbb{E}_{\mathscr{P}_T^Y}\left[F_S \prod_{i=1}^m \kappa_{\Gamma_i}^Y \right] = \mathbb{E}_{\mathscr{P}}\left[F_S \prod_{i=1}^m \kappa_{\Gamma_i} \right] \tag{6.56}$$

uniformly in Y, and we also have

$$\left| \mathbb{E}_{\mathscr{P}_T^Y}\left[F_S \prod_{i=1}^m \kappa_{\Gamma_i}^Y \right] \right| \leqslant \max |F_S| \prod_{i=1}^m 2^{|\Gamma_i|} |K_{\Gamma_i}| \tag{6.57}$$

By (6.57) we then obtain

$$\begin{aligned} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_m\} \\ \Gamma_j^* \cap S^* = \varnothing,\, j=1,\dots,m}} \prod_{i=1}^m 2^{|\bar{\Gamma}_i|} |K_{\Gamma_i}|\, 2^{|\bigcup_i \bar{\Gamma}_i|} 2^{|S^*|} &\leqslant 2^{|S^*|} \sum_{m=1}^{|S^*|} \binom{|S^*|}{m} \left(\sum_{\Gamma : \Gamma \ni 0} 4^{|\bar{\Gamma}|} |K_\Gamma| \right)^m \\ &\leqslant 2^{|S^*|} \sum_{m=1}^{|S^*|} \binom{|S^*|}{m} \left(\sum_{n=2}^{\infty} 4^n c \eta(\lambda)^n \right)^m \\ &< \infty \end{aligned}$$

Here S^* denotes the time points occurring in S. By using now this estimate together with (6.56), and applying Lebesgue's dominated convergence theorem once again we arrive to

$$\mathbb{E}_{\mathscr{P}}[F_S]\, f_S + \sum_{m=1}^{\infty} \sum_{\substack{\{\Gamma_1,\ldots,\Gamma_m\} \\ \Gamma_i^* \cap S^* \neq \emptyset,\, i=1,\ldots,m}} \mathbb{E}_{\mathscr{P}}\left[F_S \prod_{i=1}^{m} \kappa_{\Gamma_i}\right] f_{S \cup (\bigcup_i \bar{\Gamma}_i)} = \mathbb{E}_{\mu}[F_S] \tag{6.58}$$

This completes the proof of the theorem. ∎

ACKNOWLEDGMENTS

J.L. is supported by Deutsche Forschungsgemeinschaft within Schwerpunktprogramm "Interagierende stochastische Systeme von hoher Komplexität." R.A.M. thanks Zentrum Mathematik of Technische Universität München for warm hospitality and financial support. He also thanks the Russian Fundamental Research Foundation (Grant 99-01-00284) for financial support. We both thank Carlo Boldrighini for kind hospitality at Università La Sapienza in Rome, where part of the manuscript has been prepared. J.L. is grateful also to Enzo Olivieri for discussions and an additional travelling grant to Università Tor Vergata, Rome. Valuable discussions with Herbert Spohn and Volker Betz are also gratefully acknowledged.

REFERENCES

1. V. Betz and J. Lőrinczi, A Gibbsian description of $P(\phi)_1$-processes (1999), submitted for publication.
2. V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos, and H. Spohn, Ground state properties of a particle coupled to a scalar quantum field—A Gibbs measure-based approach, TU München preprint, 2000.
3. M. Cassandro and E. Olivieri, Renormalization group and analyticity in one dimension: A proof of Dobrushin's theorem, *Commun. Math. Phys.* **80**:255–269 (1981).
4. E. B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, *J. Func. Anal.* **59**:335–395 (1984).
5. R. L. Dobrushin, Analyticity of correlation functions in one-dimensional classical systems with slowly decreasing potentials, *Commun. Math. Phys.* **32**:269–289 (1973).
6. W. G. Faris and R. A. Minlos, A quantum crystal with multidimensional anharmonic oscillators, *J. Statist. Phys.* **94**:365–388 (1999).
7. H. O. Georgii, *Gibbs Measures and Phase Transitions* (Walter de Gruyter, 1988).
8. J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View* (Springer, 1981).
9. Y. Hariya and H. Osada, Diffusion processes on path spaces with interactions, preprint, 2000.

10. C. Kipnis and S. R. S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications for simple exclusions, *Commun. Math. Phys.* **104**:1–19 (1986).
11. B. Ya. Levin, *Distribution of Roots of Entire Functions* (Gostechisdat, Moscow, 1956) (in Russian).
12. J. Lőrinczi, R.A. Minlos, and H. Spohn, The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field, 2000 (submitted for publication).
13. V. A. Malyshev and R. A. Minlos, *Gibbs Random Fields* (Kluwer Academic Publishers, 1991).
14. R. A. Minlos, A. Verbeure, and V. Zagrebnov, A quantum crystal model in the light-mass limit: Gibbs state, *Rev. Math. Phys.* **12**:981–1032 (2000).
15. E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5**:1190–1197 (1964).
16. H. Osada and H. Spohn, Gibbs measures relative to Brownian motion, *Ann. Probab.* **27**:1183–1207 (1999).
17. B. Simon, *Functional Integration and Quantum Physics* (Academic Press, 1979).
18. B. Simon, Schrödinger semigroups, *Bull. AMS* **7**:447–526 (1982).
19. E. Zhizhina, Yu. Kondratiev, and R. A. Minlos, The lower branches of the spectrum in the Hamiltonian of infinite quantum systems with compact space of "spins," *Trudy Moscow Math. Soc.* **60**:289–302 (1999).